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NON-SQUARE DETERMINANTS AND MULTILINEAR VECTORS

H. RANDOLPH PYLE, Whittier College

When a student learns of the many relationships between determinants and square matrices, he is apt to wonder why there isn't a theory of non-square determinants to go with that of non-square matrices. As a matter of fact it is quite possible to present such a theory as we shall see. In order to keep the length of the presentation to a minimum we shall base our discussion on that of a standard modern text, namely Chapter 5 of "Elements of Linear Algebra" by L. J. Paige and J. D. Swift [1].

Vectors in the real vector space $V_n(R)$ are defined as sets of ordered n-tuples of real numbers, $X = (x_1, x_2, \cdots, x_n)$. A determinant $D(X_1, X_2, \cdots, X_n)$ is defined as a real number assigned to the ordered set of n vectors $X_1, X_2, \cdots, X_n$, which has the following properties: (1) D is linear in each of its arguments, (2) D is zero if any two of its arguments are equal, (3) $D = 1$ when $X_i = E_i$, where E_i is the natural basis vector whose ith component is one and whose other components are zero.

The classical properties of determinants are developed from this definition. The fact that we have n vectors, each with n components, gives us the square array of the determinant.

Now suppose that we revise our definition so that we have r vectors $(r \leqq n)$ instead of n vectors, and reword our definition as follows:

Definition: A determinant $D(X_1, \cdots, X_r)$, $r \leqq n$, is an ordered set of $\binom{n}{r}$ real numbers corresponding to the ordered set of r vectors $X_1, \cdots, X_r$ in $V_n(R)$, which has the properties listed above.

This ordered set of $\binom{n}{r}$ real numbers consists of the numbers associated with each of the $\binom{n}{r}$ square r-rowed determinants that can be formed by selecting r columns from the array of vectors, arranged in a predetermined order. For example, if $n = 3$, $r = 2$, and $X_i = (x_{i1}, x_{i2}, x_{i3})$, the (2×3)-determinant

$$D(X_1, X_2) = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{vmatrix} = \left(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}, \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix}, \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} \right)$$

Since each of the components of the $(r\times n)$-determinant involves the r rows of the determinant, any valid operation on the rows of a square determinant is equally valid for each component of the non-square determinant. For instance, if two rows are permuted the sign of each component of D is changed and we can say that D changes sign. If the row vectors are linearly dependent, each component of D vanishes and we say that $D = 0$. Conversely, if $D = 0$, the vectors are dependent. If every element of a row is zero, or if two rows are proportional, $D = 0$. If a multiple of one row is added to another row, the value of the determinant is not changed.

But it is no longer true that the transpose of a determinant equals the determinant. The usual theorem that the roles of columns and rows may be interchanged no longer holds. Instead we may say that any operation valid for the rows of a determinant is valid for the columns of its transpose.

In defining equality and the sum, product and magnitude of non-square determinants we may treat them as vectors in an Euclidean space of $\binom{n}{r}$ dimensions. Two non-square determinants are equal when their corresponding components are equal.

If

$$D_1(X_1, \cdots, X_r) = (u_1, u_2, \cdots, u_k),$$

$k = \binom{n}{r}$, and

$$D_2(Y_1, \cdots, Y_r) = (v_1, v_2, \cdots, v_k),$$

we define the sum of D_1 and D_2 as a new vector whose components are the sums of corresponding components of D_1 and D_2, i.e.,

$$D_1 + D_2 = (u_1 + v_1, u_2 + v_2, \cdots, u_k + v_k).$$

The product of D_1 and D_2 is defined as the vector inner product, i.e., the sum of the products of corresponding components. We shall designate it by $D_1 \cdot D_2$, so that

$$D_1 \cdot D_2 = u_1 v_1 + u_2 v_2 + \cdots + u_k v_k.$$

The magnitude of D_1 is defined as the square root of the sum of the squares of its components, so that

$$|D_1| = (D_1 \cdot D_1)^{1/2} = (u_1^2 + u_2^2 + \cdots + u_k^2)^{1/2}.$$

An alternative approach to the product is to use matrix multiplication. If we designate the corresponding matrices as

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_r \end{pmatrix}$$

where the X_i, Y_i are the row vectors of the matrices, the transpose of Y is $Y^t = (Y_1^t, Y_2^t, \cdots, Y_r^t)$, where the Y_i^t are the column vectors which are transposes of the row vectors of Y. The matrix product XY^t is square and its determinant is an orthodox one. It can be shown [2] that this determinant has the same value as $D_1 \cdot D_2$ already defined. That is

$$D_1 \cdot D_2 = u_1 v_1 + u_2 v_2 + \cdots + u_k v_k = \begin{vmatrix} X_1 \cdot Y_1 & X_1 \cdot Y_2 & \cdots & X_1 \cdot Y_r \\ X_2 \cdot Y_1 & X_2 \cdot Y_2 & \cdots & X_2 \cdot Y_r \\ & \ddots & & \\ X_r \cdot Y_1 & X_r \cdot Y_2 & \cdots & X_r \cdot Y_r \end{vmatrix}$$

The latter form of the product is frequently best for computation.

When $X_i = Y_i$, $D_1 = D_2$ and the magnitude of D is

$$|D| = (D \cdot D)^{1/2} = \begin{vmatrix} X_1 \cdot X_1 & X_1 \cdot X_2 & \cdots & X_1 \cdot X_r \\ & \cdots & & \\ X_r \cdot X_1 & X_r \cdot X_2 & \cdots & X_r \cdot X_r \end{vmatrix}^{1/2}.$$

Content of a parallelotope. Paige and Swift define the content (volume) of a parallelotope with the n edges $X_1, \cdots, X_n$ in Euclidean space of n dimensions as the magnitude of the determinant $D(X_1, \cdots, X_n)$. We can generalize this to say that the content of a parallelotope with r edges $X_1, \cdots, X_r$ in Euclidean n-space ($r \leqq n$) is equal to the magnitude of the non-square determinant $D(X_1, \cdots, X_r)$. This reduces to the usual formulas in two and three dimensions. For instance, the area of a parallelogram in 3-space whose edges are the vectors X_1, X_2 may be found by taking the magnitude of the cross product of X_1 and X_2. When $X_i = (x_{i1}, x_{i2}, x_{i3})$,

$$X_1 \times X_2 = \left(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}, \begin{vmatrix} x_{13} & x_{11} \\ x_{23} & x_{21} \end{vmatrix}, \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} \right).$$

Except for the sign of the middle term, these components are the same as those of $D(X_1, X_2)$ so that $|D(X_1, X_2)|$ gives the area of the parallelogram.

Multilinear vectors. Grassmann, Cartan, Bourbaki and other writers have named these parallelotopes multilinear vectors. Cartan [3] calls a parallelogram a bivector and a parallelepiped a trivector. They represent the outer product of the line vectors that form their edges. He uses brackets $[X_1 X_2 \cdots X_r]$ for this product and calls it an r-vector. It is best represented by an $(r \times n)$-determinant. The components of the determinant are the coordinates of the r-vector in n-space. The magnitude of the r-vector is the magnitude of the determinant. Two r-vectors are equal when their determinants are equal. The sum of two such vectors is the sum of their determinants and their scalar product is the product of the determinants. This scalar product is the magnitude of the first times the projection of the second on the space spanned by the first. If we designate the r-vectors by $A_{(r)}$ and $B_{(r)}$, where the subscript in parentheses represents the dimension of the vector, and write their scalar product as $A_{(r)} \cdot B_{(r)}$, we define $A_{(r)} \cdot B_{(r)} = |A_{(r)}||B_{(r)}| \cos \theta$, where θ is the angle between the two r-flats. This serves to define θ.

Two r-vectors are said to be orthogonal when their inner product is zero. This means that at least one line vector in each is perpendicular to every line vector in the other. To show this let $L = x_1 B_1 + x_2 B_2 + \cdots + x_r B_r$, where the B_i are line vectors defining $B_{(r)}$ and the x_i are scalars. If L is orthogonal to every vector in $A_{(r)}$, $A_i \cdot L = 0$, $(i = 1, \cdots, r)$. This gives a set of r homogeneous equations in the r variables $x_1, \cdots, x_r$, whose coefficients are of the form $A_i \cdot B_j$. These equations will have a non-trivial solution if and only if the determinant of the coefficients vanishes. But this determinant is $A_{(r)} \cdot B_{(r)}$, so that the vanishing of the inner product assures such a vector.

When every line vector in $B_{(r)}$ is perpendicular to every line vector in $A_{(r)}$, we say that the two r-vectors are completely orthogonal. Then every element in the determinant of the product vanishes.

Reciprocal multiple vectors and determinants. Greville [4] discusses the pseudoinverse, as he calls it, of a non-square matrix. If we have an r-vector $A_{(r)} = [A_1 A_2 \cdots A_r]$, the set of line vectors A_i^* reciprocal to the vectors A_i with respect to the r-flat define an r-vector $A_{(r)}^* = [A_1^* A_2^* \cdots A_r^*]$, whose matrix turns out to be the pseudoinverse of Greville. In vector terms $A_{(r)} \cdot A_{(r)}^* = 1$, and $A_{(r)}$ and $A_{(r)}^*$ are reciprocal vectors. We may call their determinants reciprocal $(r \times n)$-determinants.

$A_{(r)}^*$ is easily found by matrix methods. If $(A_{(r)}^2)^{-1}$ is the inverse of the square matrix corresponding to $A_{(r)} \cdot A_{(r)}$, then using matrix multiplication, $A_{(r)}^* = (A_{(r)}^2)^{-1} A_{(r)}$.

There are many interesting relations in the theory of multilinear vectors, but these are enough to show how they relate to non-square determinants.

Relations between the components of an $(r \times n)$-determinant. We have said that the $\binom{n}{r}$ components of an $(r \times n)$-determinant resemble the components of a vector in a space of $\binom{n}{r}$ dimensions. This is true but there are identities between the components that limit the number of independent ones. In order to find these identities, let us consider

$$D(X_1, \cdots, X_r) = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ & & \cdots & \\ x_{r1} & x_{r2} & \cdots & x_{rn} \end{vmatrix}, \quad r \leqq n .$$

If we form an $(r+1)$-rowed determinant by putting any one of the vectors $X_1, \cdots, X_r$ at the top of the existing determinant, the new matrix is still of rank r (assuming that the original vectors are independent) because the new vector is a repetition of one of the old ones. This means that every $(r+1)$-rowed square determinant in the array vanishes. Expanding by the elements of the top row gives an identity relation between the elements of the vector in that row and some of the components of $D(X_1, \cdots, X_r)$.

There will be $\binom{n}{r+1}$ identities for each vector added at the top and r

such vectors that may be used, making a total of $r\binom{n}{r+1}$ identities. This procedure can be repeated with two rows added at the top, giving $\binom{r}{2}\binom{n}{r+1}$ identities, involving the components of $D(X_1, \cdots, X_r)$ and the 2-rowed minors of the elements in the top two rows. This process can be repeated until we add r rows.

When $n \geqq 2r$ we can add all r vectors of the original determinant. Then the identities can be expressed in terms of the components of D only. There will be $\binom{n}{2r}$ of them. For example, when $n = 4$, $r = 2$,

$$D(X_1, X_2) = \begin{vmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{vmatrix} = (u_{12}, u_{13}, u_{14}, u_{23}, u_{24}, u_{34})$$

where the subscripts of the u's indicate the columns involved in the component determinants. The augmented determinants are

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{vmatrix}, \quad \begin{vmatrix} x_{21} & x_{22} & x_{23} & x_{24} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{vmatrix}, \quad \begin{vmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{vmatrix}$$

From the first of these we get the identities :

$$x_{11}u_{23} - x_{12}u_{13} + x_{13}u_{12} = 0 \qquad x_{11}u_{34} - x_{13}u_{14} + x_{14}u_{13} = 0$$

$$x_{11}u_{24} - x_{12}u_{14} + x_{14}u_{12} = 0 \qquad x_{12}u_{34} - x_{13}u_{24} + x_{14}u_{23} = 0 \, .$$

From the second we get four more derived from these by replacing x_{1i} by x_{2i}. From the last determinant we have

$$u_{12}u_{34} - u_{13}u_{24} + u_{14}u_{23} = 0 \, .$$

Conclusion. When presented in this fashion the theory of non-square determinants seems to follow naturally from that of square determinants. Their applications are useful extensions of those of square determinants.

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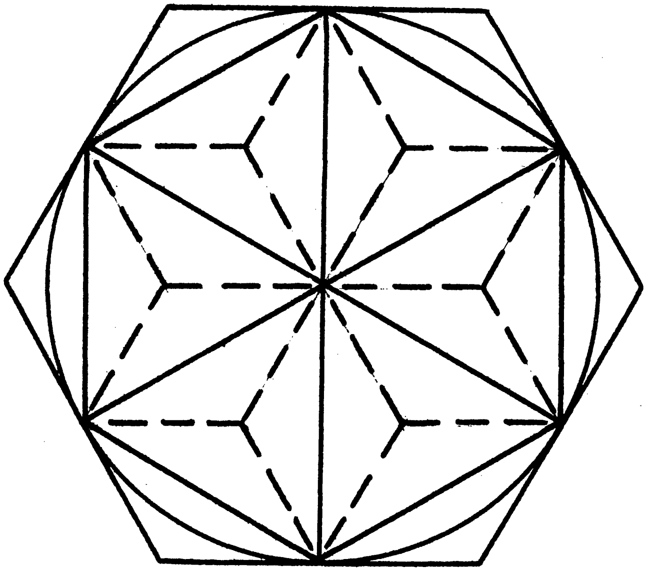
A HEXAGONAL CONFIGURATION

CHARLES W. TRIGG, Los Angeles City College



Consider an inscribed hexagon with vertices at the midpoints of the circumscribed hexagon. By drawing the radii of the inscribed hexagon, and then joining the centroids of the six equilateral triangles thus formed to the vertices of their respective triangles, the configuration is dissected into 24 congruent triangles. Eighteen of these lie in the inscribed hexagon, so the areas of the inscribed and circumscribed hexagons are in the ratio of 18 : 24 or 3 : 4.

Taken by twos, the triangles form 12 congruent rhombii. The configuration is a segment of a space-filling tessellation of rhombii. The six rhombii with vertices at the center of the hexagons constitute a 6-pointed star, which has one-half the area of the circumscribed hexagon.

Embedded in the configuration is the familiar optical illusion of three stepped cubes. The top cube may be chosen in six ways.

POINTS OF POLYGONAL REGIONS

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Introduction. This paper gives a solution to the problem of determining if a point on the earth's surface is inside, on the boundary, or outside of a polygonal region on the earth. The point and the polygonal region are assumed to be in the northern hemisphere. The earth is assumed to be spherical.

Solution to Problem. By using the polar gnomonic projection, which projects points on the surface of the earth from the earth's center onto the tangent plane at the North Pole, we reduce the problem to the equivalent problem of determining if an arbitrary point is inside, on the boundary, or outside of a polygonal region in the plane.

Let $P_1, P_2, P_3, \cdots, P_n$ denote the sequence of vertices defining the polygon and Q denote an arbitrary point. Clearly, Q is on a vertex of the polygon if it has the same coordinates as some P_i. Also, Q is on a side of the polygon if the vector equation

$$\overrightarrow{QP_i} \times \overrightarrow{QP_{i+1}} = 0$$

holds true for some $1 \leqq i \leqq n$ with, of course, the agreement that $P_{n+1} = P_1$.

Since a polygon consists of only vertices and sides, these two tests are sufficient to determine if Q is on the boundary of the polygonal region and, hence, give a solution to a portion of the problem. The remaining part of the problem is to determine if Q is inside or outside of the polygonal region. The solution uses the fact that if a point is inside a polygonal region then a half-line originating at this point and not passing through any vertex intersects the sides of the polygon an odd number of times.

This fact may be verified as follows: Assume that Q is inside the polygonal region and let $x_1, x_2, x_3, \cdots, x_n$ be the sequence of n successive points of intersection of the sides of the polygon with a half-line originating at Q. Since the region is bounded there must be at least one point of intersection. From the open intervals (x_i, x_{i+1}) we form the correspondence $(Q, x_1) \to$ inside, $(x_1, x_2) \to$ outside, $(x_2, x_3) \to$ inside, etc.; which shows that $(x_n, \infty) \to$ inside holds true if n is even and $(x_n, \infty) \to$ outside if n is odd. But, since the polygonal region is bounded, $(x_n, \infty) \to$ inside is impossible and, hence, n must be odd. By this argument one may show also that if Q is outside the region then a half-line originating at Q and not passing through any vertex intersects the sides of the polygon an even number of times or not at all.

Describing the half-line through Q by its unit tangent vector $\vec{U}$, it follows that a half-line originating at Q and not passing through any vertex intersects side P_iP_{i+1} of the polygon if and only if the vector equation

$$t_1\vec{U} = \overrightarrow{QP}_i + t_2(\overrightarrow{QP}_{i+1} - \overrightarrow{QP}_i)$$

holds true for $t_1 > 0$ and $0 < t_2 < 1$. Since the vectors in this equation are in the same plane, the equation designates two equations in two unknowns t_1 and t_2 and, therefore, can be readily solved. Thus, if Q is not a boundary point, it is inside the polygonal region if and only if $t_1 > 0$ and $0 < t_2 < 1$ satisfy the above equation for an odd number of sides.

Acknowledgement. The author wishes to thank Dr. Karl Folley for his helpful suggestions.

CORRECTION

S.K. Chatterjea, *On Congruence Properties of Legendre Polynomials*, this Magazine, Sept.-Oct., 1961, pp. 329-336, the congruence (1.1) should be

$$P_n \equiv P_{a_0} P_{a_1}^{p} P_{a_2}^{p^2} \cdots P_{a_r}^{p^r} \pmod{p}$$

where

$$n = a_0 + a_1 p + a_2 p^2 + \cdots + a_r p^r \quad (0 \leqq a_i < p) .$$

In (4.8) we should have

$$d_2 = 2^2 c_2 + 2c_1 \binom{2m-1}{1} .$$

(4.11) should be

$$= d_m (x^2-1)^{mp} + \sum_{i=1}^{m-1} d_i (x^2-1)^{ip} \{(x-1)^{(2m-2i)p} + (x+1)^{(2m-2i)p}\} .$$

In (4.12) we should have

$$d_m = 2^m c_m + \sum_{j=1}^{m-1} 2^j c_j \binom{2m-2j}{m-j} .$$

In Theorem 2, the last equation for determining the constants c_i, should be

$$\binom{2m}{m} \left\{ \binom{2m}{m} - 1 \right\} = 2^m c_m + \sum_{j=1}^{m-1} 2^j c_j \binom{2m-2j}{m-j} .$$

In page 336, in the last line we should have $d_1 = -\frac{1}{2} k(k-1)$.

ON THE RANK OF A MATRIX

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In a recent note [1], Murthy has given a very simple proof of the well-known theorem concerning the rank of the product of two matrices. This proof is based on the concept of linear dependence of vectors. However, the proof also requires use of the fact that the row rank and column rank of a matrix are the same. The purpose of this note is to give an elementary proof of the theorem in [1], which does not make use of the equivalence of row and column rank, but rather proves these results simultaneously. In addition, we prove that the rank of a matrix equals the rank of its corresponding Gram matrix, and conclude by giving two further definitions of the rank of a matrix.

We shall assume only the following simple facts concerning matrices and determinants:

1°. A system of n homogeneous linear equations in n unknowns has a non-trivial solution if, and only if, its determinant of coefficients is zero.

2°. If A is an $n \times n$ matrix, A' its transpose, and $|A|$ denotes the determinant of A, then $|A| = |A'|$.

3°. If A, B are two matrices for which AB is defined, then $(AB)' = B'A'$.

Now, let A be an $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} = (\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n) .$$

DEFINITION 1. *By the row-rank, $r_R(A)$, of a matrix A we mean the maximum number of linearly independent row-vectors α_i. Similarly, $r_C(A)$, the column-rank of A denotes the maximum number of linearly independent column-vectors $\bar{\alpha}_j$.*

We present our preliminary results in the form of a series of statements numbered I, II, etc.

I. *If A is an $n \times n$ matrix, then $|A| \neq 0$ if, and only if, $r_R(A) = n$.*

For, consider the n homogeneous linear equations

$$c_1 a_{1j} + c_2 a_{2j} + \cdots + c_n a_{nj} = 0 \qquad (j = 1, 2, \cdots, n)$$

in the n unknowns $c_1, \cdots, c_n$. By 1° these equations have a non-trivial solution if, and only if, $|A'| = 0$, or, according to 2° if, and only if, $|A| = 0$. By definition 1, on the other hand, they have a non-trivial solution if, and only if, $r_R(A) < n$. Since $r_R(A) \leqq n$, the result now follows.

I′. *If A is an $n \times n$ matrix, then $|A| \neq 0$ if, and only if, $r_C(A) = n$.*

This follows from I by a simple duality argument by setting $A' = B$, and noting that $r_C(A) = r_R(B)$, so that

$$r_C(A) = n \iff r_R(B) = n \iff |B| \neq 0 \iff |A| \neq 0 .$$

II. *If* $\tilde{A}$ *is any submatrix of* A, *then* $r_R(\tilde{A}) \leqq r_R(A)$.

To prove this, suppose that $r_R(\tilde{A}) = s$; then A has s linearly independent row-vectors, say $\tilde{\alpha}_{i_1}, \tilde{\alpha}_{i_2}, \cdots, \tilde{\alpha}_{i_s}$, where

$$\tilde{\alpha}_{i_j} = (a_{i_j k_1}, a_{i_j k_2}, \cdots, a_{i_j k_t}) ,$$

and $k_1, \cdots, k_t$ is a subset of $1, 2, \cdots, n$. We now assert that the row-vectors $\alpha_{i_1}, \cdots, \alpha_{i_s}$ of A are themselves linearly independent, and hence that $r_R(A) \geqq s = r_R(\tilde{A})$. For, suppose that

$$c_1\alpha_{i_1} + c_2\alpha_{i_2} + \cdots + c_s\alpha_{i_s} = \vec{0} ;$$

then

$$c_1 a_{i_1 j} + c_2 a_{i_2 j} + \cdots + c_s a_{i_s j} = 0 \qquad (j = 1, 2, \cdots, n) .$$

In particular, on selecting those equations for which $j = k_1, \cdots, k_t$, we have

$$c_1\tilde{\alpha}_{i_1} + c_2\tilde{\alpha}_{i_2} + \cdots + c_s\tilde{\alpha}_{i_s} = \vec{0} .$$

It follows that $c_1 = c_2 = \cdots = c_s = 0$, proving our assertion.

II′. If $\tilde{A}$ *is any submatrix of* A, *then* $r_C(\tilde{A}) \leqq r_C(A)$.

For, if $B = A'$, and $\tilde{B} = \tilde{A}'$, then $\tilde{B}$ is a submatrix of B and hence

$$r_C(\tilde{A}) = r_R(\tilde{B}) \leqq r_R(B) = r_C(A) .$$

III. *For matrices of the proper dimensions,* $r_R(AB) \leqq r_R(A)$.

This is precisely what was proved in [1]. For completeness, we include here what is essentially the same proof. Let A be an $m \times n$ matrix with row-vectors $\alpha_1, \cdots, \alpha_m$, and B be an $n \times r$ matrix with column-vectors $\overline{\beta}_1, \cdots, \overline{\beta}_r$. Then

$$AB = \begin{pmatrix} \alpha_1 \cdot \overline{\beta}_1 & \alpha_1 \cdot \overline{\beta}_2 & \cdots & \alpha_1 \cdot \overline{\beta}_r \\ \alpha_2 \cdot \overline{\beta}_1 & \alpha_2 \cdot \overline{\beta}_2 & \cdots & \alpha_2 \cdot \overline{\beta}_r \\ \vdots & \vdots & & \vdots \\ \alpha_m \cdot \overline{\beta}_1 & \alpha_m \cdot \overline{\beta}_2 & \cdots & \alpha_m \cdot \overline{\beta}_r \end{pmatrix}$$

where $\alpha_i \cdot \overline{\beta}_j$ denotes the scalar or dot product of the two vectors involved. Now, let $r_R(AB) = q$ ($\leqq m$). Then AB contains q linearly independent row-vectors. Suppose these are the rows corresponding to subscripts

$i_1, \dots, i_q$. Then we assert that the row-vectors $\alpha_{i_1}, \dots, \alpha_{i_q}$ of A are linearly independent from which our result will follow. Indeed, if

$$c_1\alpha_{i_1} + c_2\alpha_{i_2} + \cdots + c_q\alpha_{i_q} = \vec{0}\ ,$$

then

$$c_1(\alpha_{i_1}\cdot\overline{\beta}_j) + c_2(\alpha_{i_2}\cdot\overline{\beta}_j) + \cdots + c_q(\alpha_{i_q}\cdot\overline{\beta}_j) = 0 \quad (j = 1, \dots, r)\ .$$

However, from our assumption concerning the q linearly independent rows of AB, it follows we must have $c_1 = \cdots = c_q = 0$, and our assertion is proved.

III . *For matrices of the proper dimension,* $r_C(AB) \leqq r_C(B)$.

Indeed, $r_C(AB) = r_R[(AB)'] = r_R(B'A') \leqq r_R(B') = r_C(B)$, using 3° and III.

IV. *For any matrix* A, $r_R(A) = r_R(AA') = r_C(AA')$.

First, the equality $r_R(AA') = r_C(AA')$ is trivial. Also, by III, we have $r_R(AA') \leqq r_R(A)$, so that it only remains to prove

$$(*) \qquad r_R(A) \leqq r_R(AA')\ .$$

Let $r_R(A) = s$, so that A has s linearly independent row-vectors. With no loss in generality, we may assume these are $\alpha_1, \dots, \alpha_s$. We have

$$G = AA' = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix}(\alpha_1, \dots, \alpha_r) = \begin{pmatrix} \alpha_1\cdot\alpha_1 & \alpha_1\cdot\alpha_2 & \cdots & \alpha_1\cdot\alpha_r \\ \vdots & & & \vdots \\ \alpha_r\cdot\alpha_1 & \alpha_r\cdot\alpha_2 & \cdots & \alpha_r\cdot\alpha_r \end{pmatrix}.$$

Consider the $s\times s$ submatrix $\tilde{G}$ of G whose principal diagonal consists of the elements $\alpha_i\cdot\alpha_i$, $i = 1, 2, \dots, s$. We shall prove that $r_R(\tilde{G}) = s$, whence the result (*) will follow from II. To prove this, suppose that

$$c_1(\alpha_1\cdot\alpha_i) + c_2(\alpha_2\cdot\alpha_i) + \cdots + c_s(\alpha_s\cdot\alpha_i) = 0 \quad i = 1, 2, \dots, s\ .$$

These equations may be rewritten in the form

$$(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_s\alpha_s)\cdot\alpha_i = 0 \quad i = 1, 2, \dots, s\ .$$

Now, multiplying the ith of these equations through by c_i, and adding, we obtain

$$(c_1\alpha_1 + \cdots + c_s\alpha_s)\cdot(c_1\alpha_1 + \cdots + c_s\alpha_s) = 0\ .$$

It follows that

$$c_1\alpha_1 + \cdots + c_s\alpha_s = \vec{0}\ ,$$

and hence that $c_1, \dots, c_s$ are necessarily all zero, since the α_i were linearly independent vectors. This proves that the s rows of $\tilde{G}$ are linearly independent, so that $r_R(\tilde{G}) = s$.

V. *For any matrix A,* $r_C(A) = r_C(AA') = r_R(AA')$.

First, note that this result does *not* follow by a duality argument, which would merely give

$$r_C(A) = r_R(A') = r_R(A'A) .$$

To prove the result, let $r_C(A) = p$. Proceed as in IV to consider the $p \times p$ principal submatrix $\tilde{G}$ of $G = AA'$. Due to the symmetry of this matrix we conclude, precisely as in IV, that $r_C(\tilde{G}) = p$. Hence

$$r_C(A) \leqq r_C(AA') .$$

On the other hand, by III$'$, we have $r_C(AA') \leqq r_C(A') = r_R(A)$, so that

$$r_C(A) \leqq r_R(A)$$

holds for any matrix A. Applying this inequality to the matrix A' gives

$$r_R(A) \leqq r_C(A) .$$

Combining inequalities the stated result follows.

Moreover, the preceding propositions have proved the following:

THEOREM 1. *For any matrix A,* $r_R(A) = r_C(A)$ *and hence we may speak of the rank of a matrix A, denoted by* $r(A)$. *For matrices A, B of the proper dimensions we have* $r(AB) \leqq \min\{r(A), r(B)\}$. *Finally, we have* $r(A) = r(AA') = r(A'A)$ *for every matrix A.*

If

$$A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}$$

is an $m \times n$ matrix, we define the *range* of A to be the set of all $1 \times n$ row-vectors Y of the form

$$Y = XA = (x_1, x_2, \cdots, x_m) \cdot A .$$

THEOREM 2. *The range of A is a vector space, and the dimension of this vector space is* $r(A)$.

To see this, note that any vector XA can be written in the form

$$XA = \{x_1(1, 0, 0, \cdots, 0) + x_2(0, 1, 0, \cdots, 0) + \cdots + x_r(0, 0, \cdots, 1)\} \cdot A$$

$$= x_1\alpha_1 + x_2\alpha_2 + \cdots + x_r\alpha_r .$$

Conversely, any linear combination of the vectors $\alpha_1, \cdots, \alpha_r$ is clearly in the range of A. Hence the range of A is the vector space generated (or spanned) by the row-vectors of A, and the dimension of this space equals the number of linearly independent α_i, namely $r(A)$.

DEFINITION 2. *The determinant rank of a matrix A is the maximum of the orders of all square submatrices $\tilde{A}$ of A for which $|\tilde{A}| \neq 0$, and is denoted by $r_D(A)$.*

THEOREM 3.

$$r_D(A) = r(A) .$$

For, if $r_D(A) = p$, then A has a $p \times p$ submatrix $\tilde{A}$ with $|\tilde{A}| \neq 0$. Hence, by I and II, $r(\tilde{A}) = p$, and $r(A) \geqq r_D(A)$.

On the other hand, if $r(A) = s$, then A has s linearly independent row-vectors $\alpha_{i_1}, \dots, \alpha_{i_s}$. Let

$$A_1 = \begin{pmatrix} \alpha_{i_1} \\ \vdots \\ \alpha_{i_s} \end{pmatrix}.$$

Then $r(A_1) = s$, and hence A_1 has s linearly independent column-vectors. Selecting these columns we obtain an $s \times s$ matrix $\tilde{A}$ which is a submatrix of A, and has s linearly independent columns. Thus $r(\tilde{A}) = s$, so $|\tilde{A}| \neq 0$, and $r_D(A) \geqq s = r(A)$. This completes the proof of the theorem.

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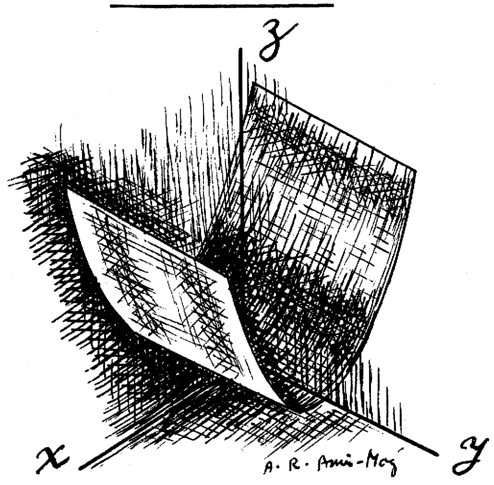

$x^2 - 2rz = 0.$

A PROPERTY OF THIRD ORDER DETERMINANTS

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If the elements of the rows of a determinant be cyclicly permuted until the elements of the first column fall along the principal diagonal, the resulting determinant is said to be *right-twisted*. Likewise, if the cyclic permutations bring the elements of the last column into position along the main cross-diagonal, the determinant is said to be *left-twisted*. Similarly, when cyclic permutation of the elements of the columns brings the elements of the first row along the principal diagonal the determinant is *down-twisted*, and when the elements of the last row fall along the main cross-diagonal the determinant is *up-twisted*.

Property. The sum of a third order determinant and its related right-twisted and left-twisted determinants is zero. The sum of a third order determinant and its related down-twisted and up-twisted determinants is zero. That is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ c_2 & a_2 & b_2 \\ b_3 & c_3 & a_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ b_2 & c_2 & a_2 \\ c_3 & a_3 & b_3 \end{vmatrix} = 0,$$

and

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_3 & c_2 \\ a_2 & b_1 & c_3 \\ a_3 & b_2 & c_1 \end{vmatrix} + \begin{vmatrix} a_1 & b_2 & c_3 \\ a_2 & b_3 & c_1 \\ a_3 & b_1 & c_2 \end{vmatrix} = 0.$$

The proof follows immediately upon expansion of the determinants.

Corollary. The sum of related third order right-twisted and left-twisted determinants equals the sum of the related up-twisted and down-twisted determinants.

SOME EXPANSIONS IN NUMBER THEORY

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The objective of this paper is to give a method for expanding a certain type of function:

$$y = f(n) ,$$

of a single variable, whose domain and range are sets of positive integers, when a characteristic function $C(u)$ is given which determines whether the integer u is in the range of f.

As an application of this method we will show how to expand the function $P(n)$ in terms of n, where $P(n)$ represents the nth prime in the set of primes:

$$2, 3, 5, 7, 11, 13, \cdots .$$

This expansion which represents P involves a finite combination of elementary functions but in the form given is not useful for the purpose of calculation by hand because of the labor involved. It is the authors' opinion that simplification is possible. Many interesting problems are posed here for the analysis expert.

We will also construct finite expansions in terms of n for the function $\pi(n)$ which represents the number of primes less than or equal to n, for $\pi'(n)$ which represents the number of pairs of twin primes (i.e. primes differing by two) whose larger member is less than or equal to n, and for $L(n)$, the least prime greater than or equal to n.

Let there be given a function $y = f(n)$ whose domain is the set of positive integers $\{n\}$ with $1 \leqq n \leqq k$, k finite or infinite, and whose range is a subset A of the set of positive integers:

$$A = \{a_i\} , \quad a_i = f(i) , \quad a_i < a_j \text{ if and only if } i < j .$$

We are also given a set B of integers:

$$B = \{b_i\} , \quad b_i \geqq a_i \text{ for all } i \text{ in the domain of } f .$$

Let the characteristic function for A be

$$C(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \notin A \end{cases} .$$

Further let

$$R(u) = \sum_{z=1}^{u} C(z)$$

be the number of integers in A each of which is less than or equal to u. Notice that $R(u) = n$ for $a_n \leqq u < a_{n+1}$, $R(u) \neq n$ for $u < a_n$ or $u \geqq a_{n+1}$.

Next let us consider the function

$$g(n, u) = \left[\frac{1}{1 + (n - R(u))^2} \right] ,$$

where $[x]$ = greatest integer $\leqq x$. Observe that

$$g(n, u) = \begin{cases} 1 & \text{if } n = R(u), \text{ i. e. if } a_n \leqq u < a_{n+1}, \\ 0 & \text{if } n \neq R(u), \text{ i. e. if } u < a_n \text{ or } u \geqq a_{n+1}. \end{cases}$$

Now to obtain the expansion for a_n note the following: the only member of the set A contained in the interval $a_n \leqq u < a_{n+1}$ is a_n so that $C(u)$ has the value 0 for all u in the interval except at a_n where $C(a_n) = 1$. Also $g(n, u)$ has the value 1 for all u in the interval and has the value 0 for all u outside the interval. Thus we see that the function $g(n, u)$ chooses the correct interval while the function $C(u)$ chooses the correct number within the interval. Accordingly we have

$$C(u) \cdot g(n, u) = \begin{cases} 1 & \text{for } u = a_n \\ 0 & \text{for } u \neq a_n \end{cases},$$

and

$$a_n = \sum_{u=k}^{b_n} (u \cdot C(u) \cdot g(n, u))$$

where $b_n \geqq a_n$ and k is any positive integer not greater than a_n. Since by hypothesis a_n is a positive integer and $a_m > a_n$ whenever $m > n$, then $a_n \geqq n$. Therefore k may be taken as n; however, in specific cases a more economical lower bound can sometimes be found. If it is difficult to find such a lower bound, then we can take $k = 1$. We now summarize in the following theorem.

THEOREM. Let $f(n)$ be a single valued function with the following properties:

1. the domain of f is the set of the first k positive integers (k finite or infinite);
2. the range of f is a set A of positive integers;
3. $f(m) > f(n)$ if and only if $m > n$;
4. a set $B = \{b_i\}$ is given with $b_i \geqq f(i)$ for all i in the domain of f;
5. a characteristic function, $C(n)$, for A is given:

$$C(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \notin A \end{cases};$$

besides $C(n)$, two other auxiliary functions are required and are defined as follows:

$$R(u) = \sum_{z=1}^{u} C(z),$$

which counts for us the number of integers in A each of which is less than or equal to u, and

$$g(n, u) = \left[\frac{1}{1+(n-R(u))^2}\right] = \begin{cases} 1 & \text{if } n = R(u), \text{ i. e. if } a_n \leqq u < a_{n+1} \\ 0 & \text{if } n \neq R(u), \text{ i. e. if } u < a_n \text{ or } u \geqq a_{n+1} \end{cases}$$

Then

$$f(n) = \sum_{u=n}^{b_n} (u \cdot C(u) \cdot g(n, u)) = \sum_{u=n}^{b_n} \left(u \cdot C(u) \cdot \left[\frac{1}{1+(n-R(u))^2}\right]\right)$$

$$= \sum_{u=n}^{b_n} \left(u \cdot C(u) \cdot \left[\frac{1}{1+(n-\sum_{z=1}^{u} C(z))^2}\right]\right)$$

where $[x]$ = the greatest integer $\leqq x$ and where the lower limit $u = n$ on the summation can be replaced by any integer less than or equal to $f(n)$.

Since $(n-R(u))^2$ is a non negative integer, the bracket function can be eliminated by any of the relations :

$$\left[\frac{1}{1+w}\right] = \frac{1-(-1)^{2^w}}{2} = \sin(2^{w-1} \cdot \pi) = \sum_{i=0}^{w} ((-1)^i \cdot P_{i+1}^{w+1} \cdot P_{w+1-i}^{w+1}),$$

where $P_s^r = \frac{\lfloor r}{\lfloor (r-s)}$, r, s, non-negative integers and $r \geqq s$ and where w is a non negative integer, and $\lfloor n$ represents factorial n.

The first two equalities are easily seen to be true since all three expressions have the value 0 if w is a positive integer, and the value 1 if $w = 0$. We will now show that

$$\left[\frac{1}{1+w}\right] = \sum_{i=0}^{w} ((-1)^i \cdot P_{i+1}^{w+1} \cdot P_{w+1-i}^{w+1}),$$

w a non negative integer.

Proof. By the binomial theorem we have

i. $$0 = (1-1)^w = \sum_{i=0}^{w} ((-1)^i \cdot C_i^w),$$

w a positive integer, $C_i^w = \frac{\lfloor w}{\lfloor w-i \cdot \lfloor i}$, the binomial coefficient. We have further

ii. $$\lfloor w \cdot C_i^w = \frac{(\lfloor w)^2}{\lfloor w-i \cdot \lfloor i} = \frac{\lfloor w}{\lfloor w-i} \cdot \frac{\lfloor w}{\lfloor i} = P_i^w \cdot P_{w-i}^w,$$

and

iii. $$(w+1)\cdot P_k^{w} = \frac{(w+1)\cdot \underline{|w}}{\underline{|w-k}} = \frac{\underline{|w+1}}{(\underline{|(w+1)-(k+1)})} = P_{k+1}^{w+1}\ .$$

Multiplying i. by $(w+1)^2\cdot\underline{|w}$ and using relations ii. and iii. we get:

$$0 = \sum_{i=0}^{w}((-1)^i\cdot(w+1)^2\cdot\underline{|w}\cdot C_i^{w}) = \sum_{i=0}^{w}((-1)^i\cdot(w+1)^2\cdot P_i^{w}\cdot P_{w-i}^{w})$$

$$= \sum_{i=0}^{w}((-1)^i\cdot(w+1)\cdot P_i^{w}\cdot(w+1)\cdot P_{w-i}^{w}) = \sum_{i=0}^{w}((-1)^i\cdot P_{i+1}^{w+1}\cdot P_{w-i+1}^{w+1})\ ,$$

for w a positive integer.

If $w = 0$,

$$\sum_{i=0}^{w}((-1)^i\cdot P_{i+1}^{w+1}\cdot P_{w-i+1}^{w+1}) = (-1)^0\cdot P_1^{1}\cdot P_1^{1} = 1\cdot 1\cdot 1 = 1\ .$$

Therefore, the expression

$$\sum_{i=0}^{w}(-1)^i\cdot P_{i+1}^{w+1}\cdot P_{w-i+1}^{w+1}$$

has the value 1 for $w = 0$ and the value 0 for w a positive integer, the same as $\left[\frac{1}{1+w}\right]$. This relation is especially interesting because it involves *only* the operations of addition, subtraction and multiplication performed upon positive integers.

We will now develop a characteristic function for prime numbers so that we can apply our theorem to obtain the expansion for the nth prime as a function of n.

Consider the function

$$h(n) = \frac{1+\underline{|n-1}}{n}\ , \quad n \text{ an integer} > 1\ .$$

By Wilson's theorem $h(n)$ is an integer if and only if n is a prime. Therefore the expression

$$\sin(h(n)\cdot\pi) = \sin\left(\frac{1+\underline{|n-1}}{n}\pi\right)$$

has the value 0 if and only if n is a prime.

For n composite we have

$$\sin\left(\frac{1+\underline{|n-1}}{n}\pi\right) = \sin\frac{\pi}{n}\cdot\cos\frac{\underline{|n-1}\cdot\pi}{n} + \cos\frac{\pi}{n}\cdot\sin\frac{\underline{|n-1}\cdot\pi}{n} = \sin\frac{\pi}{n}\ , \quad \text{for } n>4\ ,$$

since $\underline{|n-1}/n$ is an even integer for n composite and $n > 4$. Hence the expression

$$\text{(D)} \qquad C(n) = 1 - \frac{\sin (h(n) \cdot \pi)}{\sin \frac{\pi}{n}} = 1 - \frac{\sin \left(\frac{1 + \underline{|n-1}}{n} \pi\right)}{\sin \frac{\pi}{n}}, \quad n > 4,$$

has the value 1 for n prime and 0 for n composite. We define our characteristic function as follows: $C(1) = 0$, $C(2) = 1$, $C(3) = 1$, $C(4) = 0$, $C(n)$ for $n > 4$ defined by (D).

The function $R(n)$ in our expansion now becomes $\pi(n)$:

$$R(n) = \pi(n) = \sum_{z=1}^{n} C(z) = 2 + \sum_{z=5}^{n} \left(1 - \frac{\sin (\pi \cdot h(z))}{\sin \frac{\pi}{z}}\right)$$

$$= 2 + \sum_{z=5}^{n} \left(1 - \frac{\sin \left(\frac{1 + \underline{|z-1}}{z} \pi\right)}{\sin \frac{\pi}{z}}\right) = n - 2 - \sum_{z=5}^{n} \frac{\sin \left(\frac{1 + \underline{|z-1}}{z} \pi\right)}{\sin \frac{\pi}{z}}$$

which is the familiar π function in Number Theory representing the number of primes less than or equal to n.

Before continuing with our problem, it is interesting to note the function

$$Q(n) = C(n) \cdot C(n-2), \quad n \geqq 5.$$

$Q(n) = 1$ when both n and $n-2$ are primes while $Q(n) = 0$ if either or both of the numbers n and $n-2$ are composite. Hence $Q(n)$ is a characteristic function for the larger member of pairs of twin primes (primes differing by 2). Summing $Q(z)$ we obtain:

$$\pi'(n) = \sum_{z=5}^{n} Q(z) = \sum_{z=5}^{n} (C(z) \cdot C(z-2))$$

$$= 1 + \sum_{z=7}^{n} \left(\left(1 - \frac{\sin (\pi \cdot h(z))}{\sin \frac{\pi}{z}}\right)\left(1 - \frac{\sin (\pi \cdot h(z-2))}{\sin \frac{\pi}{z-2}}\right)\right)$$

$$= 1 + \sum_{z=7}^{n} \left(\left(1 - \frac{\sin \left(\frac{1 + \underline{|z-1}}{z} \pi\right)}{\sin \frac{\pi}{z}}\right)\left(1 - \frac{\sin \left(\frac{1 + \underline{|z-3}}{z-2} \pi\right)}{\sin \frac{\pi}{z-2}}\right)\right),$$

$n \geqq 7$, which represents the number of sets of twin primes whose larger member is less than or equal to n.

To continue with the expansion for $P(n)$, we still need to obtain the set $B = \{b_n\}$ with b_n greater than or equal to the nth prime. By Bertrand's postulate (proved by Tchebychef in 1850) there is at least one prime in the interval $m < x < 2m-2$, for m an integer, $m \geqq 4$. Using the weaker form that there is at least one prime in the interval $m < x < 2m$ and taking $m = 2^n$, we see that the nth prime is $\leqq 2^n$, so we take $b_n = 2^n$.

Combining these results and using our theorem we obtain an expansion, in finite form, for $P(n)$, the nth prime.

$$P(n) = \sum_{u=n}^{2^n}\left(u \cdot C(u) \cdot \left[\frac{1}{1+(n-\pi(u))^2}\right]\right) = \sum_{u=n}^{2^n}\left(u \cdot C(u) \cdot \left[\frac{1}{1+(n-\sum_{z=1}^{u} C(z))^2}\right]\right)$$

$$= \sum_{u=n}^{2^n}\left(u \cdot \left(1 - \frac{\sin\left(\frac{1+|u-1|}{u}\pi\right)}{\sin\frac{\pi}{u}}\right) \cdot \left[\frac{1}{1+\left(n-u+2+\sum_{z=5}^{u} \frac{\sin\left(\frac{1+|z-1|}{z}\pi\right)}{\sin\frac{\pi}{z}}\right)^2}\right]\right)$$

for $n > 4$.

An equivalent expansion for primes was given by one of the authors in 1950 but the proof was never published by him [1].

Another type of expansion giving the greatest or least integer in a set of integers is illustrated by the problem of finding the least prime, $L(n)$, greater than or equal to a given integer n.

If $n+k$ is the least prime not less than n and $C(n)$ is a characteristic function for primes, we have:

$$C(n+i) = \begin{cases} 0 & \text{for } 0 \leqq i < k, \text{ if } k > 0, \\ 1 & \text{for } i = k, \end{cases}$$

and

$$C'(n+i) = 1 - C(n+i) = \begin{cases} 1 & \text{for } 0 \leqq i < k, \text{ if } k > 0, \\ 0 & \text{for } i = k. \end{cases}$$

It follows at once that

$$H(n, j) = \prod_{i=0}^{j-1} C'(n+i) = \prod_{i=0}^{j-1}(1 - C(n+i)) = \begin{cases} 1 & \text{for } j \leqq k \\ 0 & \text{for } j > k \end{cases}.$$

Thus we see that as soon as the function $C(u)$ has chosen the correct number (the least prime greater than or equal to n), then the function $H(n, j)$ automatically annihilates every term thereafter. Now we write

$$Z(n+j) = C(n+j) \cdot \prod_{i=0}^{j-1} C'(n+i) = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k. \end{cases}$$

Multiplying $Z(n+j)$ by $n+j$ and summing, we obtain the expansion:

$$L(n) = n \cdot C(n) + \sum_{j=1}^{n-2} ((n+j) \cdot Z(n+j))$$

$$= n \cdot C(n) + \sum_{j=1}^{n-2} ((n+j) \cdot C(n+j) \cdot \prod_{i=0}^{j-1} C'(n+i))$$

$$= n \cdot \left(1 - \frac{\sin\left(\frac{1+\lfloor n-1 \rfloor}{n}\pi\right)}{\sin\frac{\pi}{n}}\right) + \sum_{j=1}^{n-2} \left((n+j)\left(1 - \frac{\sin\left(\frac{1+\lfloor n+j-1 \rfloor}{n+j}\pi\right)}{\sin\frac{\pi}{n+j}}\right) \prod_{i=0}^{j-1} \frac{\sin\left(\frac{1+\lfloor n+i-1 \rfloor}{n+i}\pi\right)}{\sin\frac{\pi}{n+i}}\right)$$

for $n > 4$.

The upper limit on the sum is determined by Bertrand's postulate (there is a prime in the interval $n < x < 2n-2$, for $n > 4$).

In similar fashion we can obtain an expansion for the greatest prime less than or equal to a given integer n:

$$G(n) = n \cdot C(n) + \sum_{j=1}^{[\frac{1}{2}(n-1)]} ((n-j) \cdot C(n-j) \cdot \prod_{i=0}^{j-1} C'(n-i)) .$$

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PROBABILITY FUNCTIONS FOR COMPUTATIONS INVOLVING MORE THAN ONE OPERATION

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In a previous article* the propagation of round-off error was studied for one operation with approximate numbers. Extending the method to more than one operation is not difficult in principle, but the details of keeping track of various ranges of integration are troublesome, as mentioned by Inman [3], and Woodward [4]. This paper presents a geometric device capable of handling the intricacies involved for a wide class of problems.

Problem: Given a random sample x_1, x_2, x_3 and x_4 of size four drawn from the rectangular distribution

$$f(x) = \begin{cases} 1 & 0 \leqq x \leqq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the frequency function of

$$z = \frac{(x_1 + x_2)}{(x_3 + x_4)}.$$

It is well known that $u = x_1 + x_2$ and $x = x_3 + x_4$ each possess a triangular distribution. Construct a right prism of altitude two units with the triangular distribution of x as its base and also a second right prism of altitude two units with the triangular distribution of u as its base. Place these two prisms together by letting the x-axis be perpendicular to the uv-plane at the origin and such that the positive y-axis and positive v-axis are in the opposite sense (Figure 1).

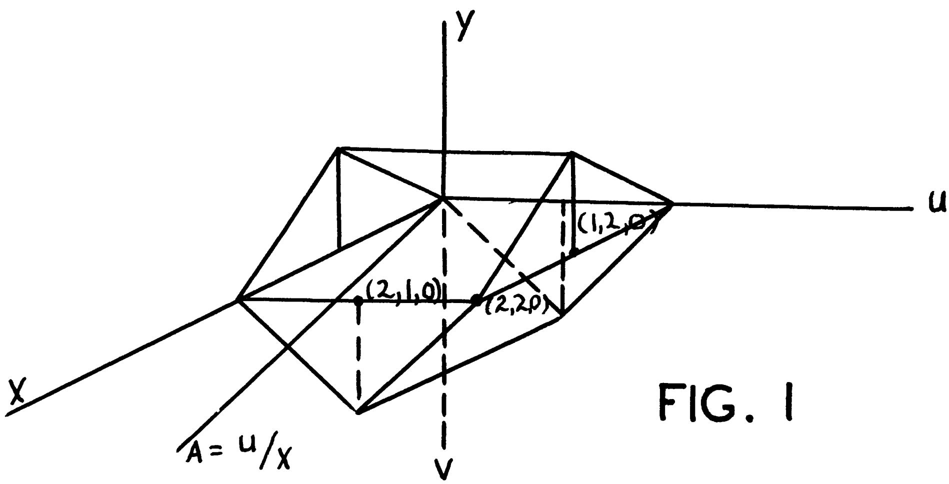

FIG. 1

For a fixed value of z, say A, the quotient $A = u/x$ represents a line

*See the *Mathematics Magazine*, Vol. 31, No. 3, Jan.-Feb., 1958.

in the xu-plane. The slope of this line will indicate the necessary divisions of the range of z values from 0 to $+\infty$; that is

$$0 \leqq z \leqq \tfrac{1}{2} \qquad \tfrac{1}{2} \leqq z \leqq 1$$
$$1 \leqq z \leqq 2 \qquad 2 \leqq z < \infty\,.$$

Define the distance functions

$$d_1(x) = \begin{cases} x & 0 \leqq x \leqq 1 \\ 2-x & 1 \leqq x \leqq 2 \end{cases}$$

and

$$d_2(u) = \begin{cases} u & 0 \leqq u \leqq 1 \\ 2-u & 1 \leqq u \leqq 1\,. \end{cases}$$

The probability that $z = u/x$ is less than some fixed value A, for $0 \leqq A \leqq \tfrac{1}{2}$, is

$$P[z < A] = \int_0^1 \int_0^{Ax} xu\, du dx + \int_1^2 \int_0^{Ax} (2-x)u\, du dx = \frac{(7A^2)}{12} + K$$

where K is a constant. Methods of elementary calculus are sufficient to write the integrals for the remaining subdivisions of our range.

Upon differentiation of this cumulative distribution with respect to A, and replacing A by the continuous variable z we have the frequency distribution;

$$g(z) = \begin{cases} \dfrac{7z}{6} & 0 \leqq z \leqq \tfrac{1}{2} \\ \dfrac{8}{3} - \dfrac{(3z)}{2} - \dfrac{2}{(3z^2)} + \dfrac{1}{(6z^3)} & \tfrac{1}{2} \leqq z \leqq 1 \\ -\dfrac{2}{3} + \dfrac{z}{6} + \dfrac{8}{(3z^2)} - \dfrac{3}{(2z^3)} & 1 \leqq z \leqq 2 \\ \dfrac{7}{(6z^3)} & 2 \leqq z < \infty\,. \end{cases}$$

When $z = (a+b)/(c+d)$, a, b, c and d represent numbers that have been rounded to the nearest integer, our method is the same. The center of our geometric solid is merely translated from the point (1, 1, 0) to $(c+d,\ a+b,\ 0)$ and we must redefine our distance functions.

Consider, for example, the quotient $z = (8-6)/(5-4)$ that has lower bound $\tfrac{1}{2}$ and is unbounded above. We must then subdivide this range of z by

$$\tfrac{1}{2} \leqq z \leqq 1 \qquad 1 \leqq z \leqq 3/2 \qquad 3/2 \leqq z \leqq 2$$
$$2 \leqq z \leqq 3 \qquad 3 \leqq z < \infty\,.$$

The two distance functions are

$$d_1(x) = \begin{cases} x & 0 \leqq x \leqq 1 \\ 2-x & 1 \leqq x \leqq 2 \end{cases}$$

and

$$d_2(u) = \begin{cases} u-1 & 1 \leqq u \leqq 2 \\ 3-u & 2 \leqq u \leqq 3 . \end{cases}$$

Omitting lengthy elementary computations, we state

$$g(z) = \begin{cases} -\frac{4}{3} + \frac{4z}{3} + \frac{1}{(3z^2)} - \frac{1}{(12z^3)} & \tfrac{1}{2} \leqq z \leqq 1 \\ \frac{13}{3} - \frac{3z}{2} - \frac{16}{(3z^2)} + \frac{11}{(4z^3)} & 1 \leqq z \leqq 3/2 \\ \frac{1}{3} - \frac{z}{6} + \frac{11}{(3z^2)} - \frac{4}{z^3} & 3/2 \leqq z \leqq a \\ -1 + \frac{z}{6} + \frac{9}{z^2} - \frac{28}{(3z^3)} & 2 \leqq z \leqq 3 \\ \frac{25}{(6z^3)} & 3 \leqq z < \infty . \end{cases}$$

We can now state the range of z as follows :

Range	Confidence level
$0.56 \leqq z \leqq 5.5$	95%
$0.51 \leqq z \leqq 13.5$	99%
$0.5 \leqq z \leqq 25$	99.9%

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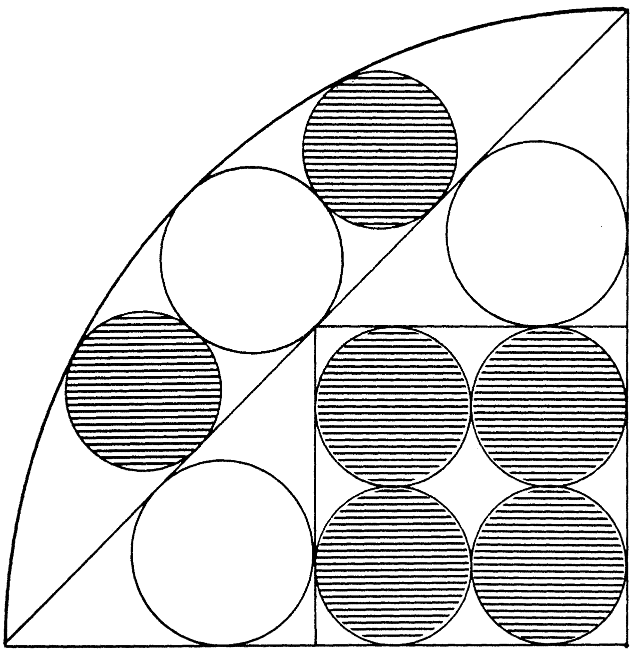
RELATED CIRCLES

LEON BANKOFF, Los Angeles, California



In the accompanying diagram, the construction of which is self-explanatory, the shaded circles are equal and the unshaded circles are equal.

PROOF. Let R denote the radius of the quadrant, r the radius of the unshaded circle inscribed in the segment, and ρ the radius of either of the shaded circles inscribed in the segment. By Johnson, *Modern Geometry*, (Dover Reprint), page 117,

$$\rho = \frac{r(R-r)}{R}.$$

Now

$$2r = R - \frac{R\sqrt{2}}{2}, \qquad \text{or} \quad r = \frac{R(2-\sqrt{2})}{4}.$$

Hence $\rho = R/8$, which corresponds to the radius of each of the shaded circles inscribed in the square. The inradius of either of the two isosceles right triangles is equal to

$$\frac{\text{(the sum of the legs minus the hypotenuse)}}{2}, \quad \text{or} \quad \frac{R(2-\sqrt{2})}{4}.$$

This corresponds to the radius of the unshaded circle inscribed in the segment.

ON A GENERALIZED FELD SERIES

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1. Introduction. The generalized Feld series [2]* discussed in this paper is designated as the Q-series and is defined as

$$Q(z) = \sum_{n=1}^{\infty} \frac{g_n(z)}{1 - f_n(z)},$$

where the sequences $\{f_n(z)\}$ and $\{g_n(z)\}$ are analytic functions in the neighborhood of the origin having value zero at the origin. Hence the Q-series also has the value zero at the origin. It can be readily seen that the Q-series reduces to Feld series, namely,

$$\sum_{n=1}^{\infty} \frac{a_n b_n z^n}{1 - a_n z^n}$$

when $g_n(z) = a_n b_n z^n$ and $f_n(z) = a_n z^n$.

The region of convergence of the Q-series, the power series representation of the Q-series and the problem of inversion of analytic functions in the neighborhood of the origin into special types of Q-series will be treated in this article.

2. Convergence. The region of existence of the Q-series is defined to be a region S such that at every point of S all of the functions $f_n(z)$, $g_n(z)$ and $g_n(z)/(1-f_n(z))$ are defined and the Q-series converges. The following theorem on the region of convergence of the Q-series is easily shown.

THEOREM. *Let I be a region such that at every point of I $|f_n(z)| < L_n$ for every n and ΣL_n converges. Let K be the region of existence of the Q-series. Then $\Sigma g_n(z)$ and $\Sigma g_n(z)/(1-f_n(z))$ converge and diverge together in $I \cap K$.*

3. Q-series as a power series. Let $f_n(z)$ and $g_n(z)$ be represented by the series

$$f_n(z) = \sum_{m=j}^{\infty} A_{nm} z^m, \quad j \geq 1,$$

and

*Numbers in brackets will refer to references at end of paper.

$$g_n(z) = \sum_{m=k}^{\infty} B_{nm} z^m , \quad k \geqq 1 ,$$

where r_n and R_n are the radii of convergence of the series respectively.

Now the power series representing each term $g_n(z)/(1-f_n(z))$ of the Q-series is the quotient of the two power series $\Sigma B_{nm} z^m$ and $(1-\Sigma A_{nm} z^m)$. We may consider it as a product of the two power series representing the functions $g_n(z)$ and $1/(1-f_n(z))$. If $\Sigma A_{nm} z^m$ is convergent in a region $|z| < r_n$ then the power series $(1-\Sigma A_{nm} z^m)$ is also convergent in that region and for $|z| \leqq t_n < r_n$, the series $(1-\Sigma A_{nm} z^m)$ is absolutely convergent. Let the zeros of the function $1-f_n(z)$ be a set $\{z\}$ such that the infimum of $\{|z|\}$ is r_n'. Then the radius of convergence of $1/(1-\Sigma A_{nm} z^m)$ is the smaller of the two numbers r_n, r_n'. Hence it follows that the power series representing the function $g_n(z)/(1-f_n(z))$ is convergent in a region K_n, where K_n consists of points within the smallest of the three concentric circles with center at the origin whose radii are R_n, r_n and r_n'. Let R_n' be the radius of K_n. Then the power series representing $g_n(z)/(1-f_n(z))$ is absolutely and uniformly convergent for $|z| \leqq \chi_n < R_n'$. Hence for $|z| \leqq \chi_n < R_n'$ we may expand $g_n(z)/(1-f_n(z))$ as a power series.

Suppose the power series of the functions $f_n(z)$ and $g_n(z)$ are

$$f_n(z) = \sum_{m=1}^{\infty} A_{nm} z^m \quad \text{and} \quad g_n(z) = \sum_{m=1}^{\infty} B_{nm} z^m$$

Let the quotient obtained from dividing $\Sigma B_{nm} z^m$ by $(1-\Sigma A_{nm} z^m)$ be denoted by

$$\sum_{m=1}^{\infty} C_{nm} z^m$$

Then the general formula for C_{nm} in terms of A_{nm}'s and B_{nm}'s is as follows:

$$(1) \qquad C_{nm} = \sum_{i=1}^{m} B_{ni} A_n^{(m+1-i)} ,$$

where

$$A_n^{(m+1-i)} = \sum_{j=1}^{N_i^{(m)}} p_{i_j}^{(m)} \cdot A_{nr_{i1}^{(j)}}^{s_{i1}^{(j)}} \cdot A_{nr_{i2}^{(j)}}^{s_{i2}^{(j)}} \cdots A_{nr_{i\chi}^{(j)}}^{s_{i\chi}^{(j)}} \cdots A_{nr_{it_j}^{(j)}}^{s_{it_j}^{(j)}} ,$$

satisfying the following conditions for the positive integers, $s_{i\chi}^{(j)}$, $r_{i\chi}^{(j)}$, t_j, $N_i^{(m)}$ and $p_{i_j}^{(m)}$:

a)
$$\sum_{\chi=1}^{t_j} s_{i\chi}^{(j)} r_{i\chi}^{(j)} = (m-i) ,$$

where $r_{i\chi}^{(j)} \neq r_{ik}^{(j)}$ for $\chi \neq k$; t_j denotes the number of distinct $r_{i\chi}^{(j)}$ in the jth term of $A_n^{(m+1-i)}$; $s_{i\chi}^{(j)}$, $r_{i\chi}^{(j)} \leqq (m-i)$; and for $j \neq k$ there exists at least one pair χ, q such that $r_{i\chi}^{(j)} \neq r_{iq}^{(k)}$ or $s_{i\chi}^{(j)} \neq s_{iq}^{(k)}$,

b) $N_i^{(m)}$ = Number of ways $(m-i)$ can be written as a sum of the form as in condition a),

c) $p_{i_j}^{(m)} = P\{r_{i\chi}^{(j)};\ s_{i\chi}^{(j)}\}_{t_j} = \dfrac{\lfloor s_{i1}^{(j)} + s_{i2}^{(j)} + \cdots + s_{it_j}^{(j)}}{\lfloor s_{i1}^{(j)} \cdot \lfloor s_{i2}^{(j)} \cdots \lfloor s_{it_j}^{(j)}}$, which is the number of permutations of the elements $r_{i1}^{(j)}, r_{i2}^{(j)}, \cdots, r_{it_j}^{(j)}$ with $r_{i\chi}^{(j)}$ repeated $s_{i\chi}^{(j)}$ times.

When $(m-i) = 0$, $A_n^{(m+1-i)} = A_n^{(1)}$. It is apparent that there are no positive integers $s_{i\chi}^{(j)}$, $r_{i\chi}^{(j)}$ satisfying conditions a), b) and c). We define $A_n^{(1)}$ as 1.

The formula (1) can be proved by induction on m. We see that C_{nm}, [4], may be expressed as

(2) $$C_{nm} = C_{n1}A_{n,m-1} + C_{n2}A_{n,m-2} + \cdots + C_{n,m-1}A_{n1} + B_{nm} .$$

Substituting (1) for C_{nk}, $k < m$, in (2) we have

(3) $$\begin{aligned} C_{nm} = {} & B_{n1}[A_n^{(1)}A_{n,m-1} + A_n^{(2)}A_{n,m-2} + \cdots + A_n^{(m-1)}A_{n1}] \\ & + B_{n2}[A_n^{(1)}A_{n,m-2} + A_n^{(2)}A_{n,m-3} + \cdots + A_n^{(m-2)}A_{n1}] \\ & + \cdots + B_{n,m-1}[A_n^{(1)}A_{n1}] + B_{nm}[A_n^{(1)}] . \end{aligned}$$

Clearly, in (3) the coefficient of $B_{n,m+1-v}$, for $v = 1, 2, \cdots, m$ is of the form

(4) $$A_n^{(v)} = A_n^{(1)}A_{n,v-1} + A_n^{(2)}A_{n,v-2} + \cdots + A_n^{(v-1)}A_{n1} .$$

Using the equation (4) it can be easily shown that the conditions a), b) and c) of formula (1) are satisfied by $A_n^{(m+1-i)}$ for every m.

If the power series of the function $g_n(z)$ is of the form

$$g_n(z) = \sum_{m=k}^{\infty} B_{nm} z^m , \quad k > 1 ,$$

we may express it as

$$g_n(z) = z^{k-1} \sum_{m=1}^{\infty} B'_{nm} z^m ,$$

where

$$B'_{nm} = B_{n,(m+k-1)} .$$

If

$$f_n(z) = \sum_{m=j}^{\infty} A_{nm} z^m , \quad j > 1 ,$$

it can be expressed as

$$f_n(z) = \sum_{m=1}^{\infty} A'_{nm} z^m ,$$

where

$$A'_{nm} = 0 \qquad \text{for } m = 1,\ 2,\ \cdots,\ (j-1)$$

and

$$A'_{nm} = A_{nm} \qquad \text{for } m = j,\ (j+1),\ \cdots$$

With these transformations for the two power series we have

$$z^{k-1} \sum_{m=1}^{\infty} C'_{nm} z^m = \frac{z^{k-1} \sum_{m=1}^{\infty} B'_{nm} z^m}{1 - \sum_{m=1}^{\infty} A'_{nm} z^m} ,$$

where C'_{nm} is obtained from (1) using B'_{nm} and A'_{nm} in place of B_{nm} and A_{nm} respectively, which in turn can be expressed as

$$z^{k-1} \sum_{m=1}^{\infty} C'_{nm} z^m = \sum_{m=1}^{\infty} C_{nm} z^m ,$$

with

$$C_{nm} = \begin{cases} 0 & , \text{ for } m \leqq k-1 \\ C'_{n,m+1-k} , & \text{for } m \geqq k . \end{cases}$$

Hence the power series representing the function $g_n(z)/(1-f_n(z))$ has the form

$$\frac{g_n(z)}{1-f_n(z)} = \sum_{m=1}^{\infty} C_{nm} z^m .$$

Let $K' = \bigcap_{n=1}^{\infty} K_n$. Then each of the power series representing the functions $g_n(z)/(1-f_n(z))$ of the Q-series is convergent in K'. So that in determining the region of convergence of the power series representing the Q-series we have to consider only the region K', where all the power series of functions $f_n(z)$, $g_n(z)$ and $g_n(z)/(1-f_n(z))$ are convergent. Applying the theorem on convergence to the Q-series with $f_n(z) = \Sigma A_{nm} z^m$ and $g_n(z) = \Sigma B_{nm} z^m$ we may state the following

COROLLARY: *Let $I' = \{z \mid |z| < r'\}$, for $z \in I'$ suppose $|\Sigma A_{nm} z^m| < L_n$ and let ΣL_n be convergent. Then $\Sigma(\Sigma B_{nm} z^m)$ and $\Sigma(\Sigma B_{nm} z^m)/(1 - \Sigma A_{nm} z^m)$ converge and diverge together in $I' \cap K'$.*

According to the theorem on uniform convergence of series [3; p. 337], the Q-series is uniformly convergent in $J' \cap I'$, where J' is the region of uniform convergence of $\Sigma g_n(z)$, and I' is a region in which $|f_n(z)| < L_n$ for every z and ΣL_n converges and I' belongs to the region of existence of the Q-series. Let W be a circular region about the origin with radius R which lies completely within $J' \cap I' \cap K'$. Then applying Weierstrass's double series theorem to the Q-series, for $z \in W$ we have

i) the series $\sum_{n=1}^{\infty} C_{nm}$ converges for every m,

ii) $Q(z) = \sum_{m=1}^{\infty} C_m z^m$ if $C_m = \sum_{n=1}^{\infty} C_{nm}$.

Hence in its region of uniform convergence the Q-series may be expanded as a power series, that is,

$$Q(z) = \sum_{n=1}^{\infty} \frac{g_n(z)}{1-f_n(z)} = \sum_{m=1}^{\infty} C_m z^m .$$

4. Inversion of power series. Let us now consider the possibility of inverting any power series representing an analytic function in the neighborhood of the origin into a Q-series. The Q-series obtained from a given analytic function will represent the same function only if the power series expansion of the Q-series and the power series representation of the given function are the same in some neighborhood of the origin.

Suppose we wish to invert a polynomial into a Q-series. A polynomial is analytic in the neighborhood of the origin and it is a formal power series with zero coefficients after a finite number of terms. We may state the following concerning the Q-series representation of a given polynomial:

Given an analytic function $P(z)$ which is a polynomial of the form

$$P(z) = \sum_{s=0}^{k} d_s z^s$$

and two double sequences of constants $\{A_{nm}\}$ and $\{C_{nm}\}$, $(n, m = 1, 2, \cdots)$ such that $\sum_{n=1}^{\infty} A_{nm}$ and $\sum_{n=1}^{\infty} C_{nm}$ are absolutely convergent and $\sum_{n=1}^{\infty} C_{nm} = d_m$, for $m = 1, 2, \cdots, k$ and for $m > k$, $C_{nm} = 0 = A_{nm}$ for all n, then in some neighborhood of the origin $P(z)$ can be represented as a Q-series, namely,

$$P(z) = d_0 + \sum_{n=1}^{\infty} \frac{g_n(z)}{1 - f_n(z)}, \tag{5}$$

where

$$f_n(z) = \sum_{m=1}^{k} A_{nm} z^m, \qquad g_n(z) = \sum_{m=1}^{2k} B_{nm} z^m,$$

where the coefficients B_{nm} are determined by (2) from the given constants A_{nm} and C_{nm}.

As we can see the series (5) is a special type of the Q-series defined by (1), since for every n, $g_n(z)$ and $f_n(z)$ are polynomials. Let another Q-series be defined as

$$Q(z) = \sum_{n=1}^{\infty} \frac{[h(z)]^n}{1 - [h(z)]^n}, \tag{6}$$

where

$$h(z) = \sum_{m=1}^{\infty} a_m z^m .$$

Then the coefficients C_m of the power series representation of (6) has the form

$$C_m = \sum_{n=1}^{\infty} C_{nm} = \sum_{m=1}^{n} \sum_{i=1}^{m} A_{ni} A_n^{(m+1-i)}, \tag{7}$$

where

$$A_{nm} = \Sigma'(a_{i_1} \cdot a_{i_2} \cdots a_{i_n}) ,$$

the sum is taken over all $(a_{i_1} \cdot a_{i_2} \cdots a_{i_n})$ such that $\sum_{j=1}^{n} i_j = m$. Hence we may state the following:

Let the power series $\sum_{m=0}^{\infty} C_m z^m$ defining an analytic function in the neighborhood of the origin be given. Then the coefficients a_m, $(m = 1, 2, \cdots)$ can be found by (7) from the coefficients C_m such that the Q-series

$$Q(z) = C_0 + \sum_{n=1}^{\infty} \frac{[h(z)]^n}{1-[h(z)]^n} ,$$

where

$$h(z) = \sum_{m=1}^{\infty} a_m z^m ,$$

represents the given power series in some neighborhood of the origin provided that

$$|\Sigma a_m z^m| < 1 , \quad \text{for some } z_0 \neq 0 .$$

Now consider as the Q-series one defined as

$$Q(z) = \sum_{n=1}^{\infty} \frac{[g(z)]^n}{1-[h(z)]^n} , \tag{8}$$

where

$$h(z) = \sum_{m=1}^{\infty} a_m z^m \quad \text{and} \quad g(z) = \sum_{m=1}^{\infty} b_m z^m .$$

In many respects the series (6) and (8) are of the same form and so with suitable conditions given we may invert any power series into a Q-series of this form. Suppose the power series expansion of (8) is of the form

$$Q(z) = \sum_{m=1}^{\infty} D_m z^m .$$

Then

$$D_m = \sum_{n=1}^{\infty} D_{nm} = \sum_{m=1}^{n} \sum_{i=1}^{m} B_{ni} A_n^{(m+1-i)} , \tag{9}$$

where

$$B_{nm} = \Sigma'(b_{i_1} \cdot b_{i_2} \cdots b_{i_n}) ,$$

the sum is taken over all $(b_{i_1} \cdot b_{i_2} \cdots b_{i_n})$ such that $\sum_{j=1}^{n} i_j = m$ and $A_{nm} = \Sigma'(a_{i_1} \cdot a_{i_2} \cdots a_{i_n})$, as in (7).

The following establishes the conditions for inverting a power series into such a Q-series :

Let the power series $\sum_{m=0}^{\infty} D_m z^m$, defining an analytic function in the neighborhood of the origin, be given. The coefficients b_m, $(m = 1, 2, \cdots)$ may be found by equation (9) from the coefficients D_m, $(m = 1, 2, \cdots)$ and a given sequence of constants a_m, $(m = 1, 2, \cdots)$. If $|\Sigma a_m z^m| < 1$, for some $z_0 \neq 0$ then $\Sigma D_m z^m$ is expressible as a Q-series of the form

$$Q(z) = D_0 + \sum_{n=1}^{\infty} \frac{[g(z)]^n}{1 - [h(z)]^n}, \tag{10}$$

where

$$h(z) = \sum_{m=1}^{\infty} a_m z^m \quad \text{and} \quad g(z) = \sum_{m=1}^{\infty} b_m z^m ,$$

in some neighborhood of the origin provided that $|\Sigma b_m z^m| < 1$, for some $z_0' \neq 0$.

Since $\Sigma g(z)^n$ is a geometric series, it converges for $|g(z)| < 1$ and diverges for $|g(z)| > 1$. According to the theorem on p. 91 the series (10) will converge for $|z| < r_0$ and diverge for $|z| > r_0$, where $r_0 = \min\{|z_0|, |z_0'|\}$. Hence we may say that (10) represents $\Sigma D_m z^m$ for every z such that $|z| < r_0$.

So far we have considered only special types of Q-series representation of analytic functions in the neighborhood of the origin. Concerning the inversion of any given power series into the general Q-series defined by (1) further studies must be done.

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A CONSTRUCTION FOR TRISECTING THE ANGLE

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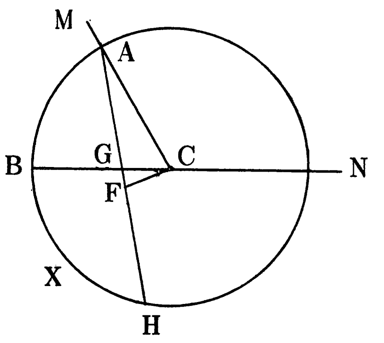
One of the famous problems of antiquity was that of trisecting the angle, that is, find a construction which uses ruler and compass only to trisect any angle. This problem resisted the efforts of all who attempted to solve it, not because our predecessors were dull, but because the problem could not be solved. In recent times the impossibility of the solution of the problem within the framework of Euclidean geometry, has been amply demonstrated.*

Nonetheless, a geometric construction for the trisection of any angle is of intrinsic interest. Of course, no construction is possible within the rigid discipline of Euclidean geometry, with its requirements that a compass be used only to draw a circle through a given point and with centre at a given point, and that a ruler be used only to draw a straight line through two given points. The only possibility of obtaining a geometric construction which will trisect any angle, is by first relaxing the above restrictions on the use of compass and ruler. We shall admit the following use of ruler and compass. We permit the ruler to pivot around a given point, at the same time the two pointers of the compass are held against the edge of the ruler, so that one pointer is moving along the arc of a given circle while the other pointer in this way has produced a curve in the plane. Note that the ruler and compass when used together in this fashion have actually constituted a distinct geometric instrument, whose use was forbidden in the Euclidean geometry. Using this instrument we shall trisect the angle.

I. To trisect any angle over 180° and under 270°.

Problem: Trisect exterior $\angle MCN$.

Construction: Construct circle with centre at C and any radius. Circle intersects line MC at A. Extend line CN through circle to B on arc.

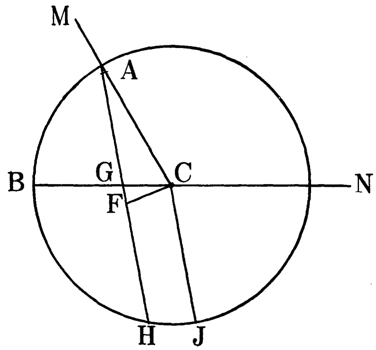


Now set compass at radius of circle, place pointers against ruler with one pointer on arc BXN. Pivot ruler about A until second pointer is on the line BC; this determines G in the line BC, and H on the arc BXN. Now

*For a proof see Courant & Robbins, *What is Mathematics?*

determine F on the line AH such that the distance AF is the radius of the circle. Join CF, then $\angle ACF$ is 1/3 exterior $\angle MCN$.

Proof: Note that A, G, F and H lie on a straight line, and that distances GH and AF are each one radius. Furthermore, ΔACH is isosceles



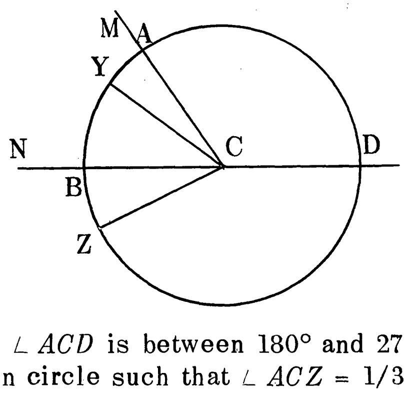
and so $\angle CAH = \angle AHC$. Hence, $\Delta ACF = \Delta HCG$. As well, both ΔACF and ΔHCG are isosceles. Now construct through C a line parallel to AH. This line intersects arc BHN at J. We shall show that $\angle NCJ = \angle JCF = \angle FCA$. First note that ΔCGF is isosceles and so $\angle CGF = \angle GFC$. Now $\angle CGF = \angle NCJ$, as well $\angle GFC = \angle JCF$, and so $\angle CGF = \angle JCF$. Finally, since ΔFCA is isosceles, $\angle AFC = \angle FCA$; but $\angle AFC = \angle GFC$, and so $\angle CGF = \angle FCA$.

Thus, $\angle NCJ = \angle JCF = \angle FCA$.

II. To trisect any angle over 0° but under 90°.

Problem: Trisect $\angle MCN$.

Construction: Construct a circle with centre at C and any radius. CM intersects circle at A, CN intersects circle at B. Line BC intersects



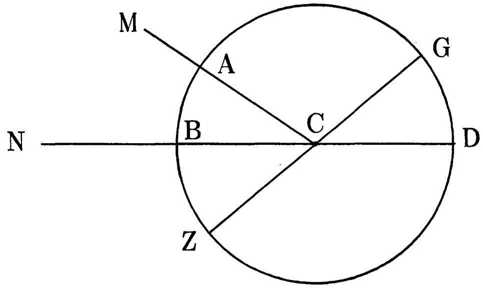
circle at D. Then ext. $\angle ACD$ is between 180° and 270° and so can be trisected. Construct Z on circle such that $\angle ACZ$ = 1/3 of ext. $\angle ACD$. With point of compass on Z, and length AC cut off Y on arc ABZ. Then $\angle ACY$ is 1/3 $\angle MCN$.

Proof: $\angle MCN$ is Ext. $\angle ACD - \angle BCD$. 1/3 $\angle MCN$ is 1/3 Ext. $\angle ACD$ − 1/3 $\angle BCD$. But BCD is a straight line and so 1/3 $\angle BCD$ is 60°. $\angle ACZ$ is 1/3 Ext. $\angle ACD$. By construction ΔYCZ is isosceles and so $\angle YCZ$ is 60°. $\angle ACZ - \angle YCZ$ is 1/3 of $\angle MCN$, i. e. $\angle ACY$ is 1/3 of $\angle MCN$.

III. To trisect any angle over 270° and under 360°.

Problem: Trisect ext. ∠ MCN.

Construction: Line MC meets circle at A, line NC meets circle at B and D. Ext. ∠ MCD is between 180° and 270° and so can be trisected.



Construct Z on circle so that ∠ ACZ is 1/3 of ext. ∠ ACD. Line ZC meets circle at G, then ∠ ACG is 1/3 of ext. ∠ MCN.

Proof:

$$\angle MCN + \text{ext. } \angle MCN = 360^\circ ,$$

$$(1/3)\,\text{ext. } \angle MCN = 120^\circ - (1/3) \angle MCN .$$

Similarly,

$$(1/3) \angle MCN = (1/3)\,\text{ext. } \angle ACD - 60^\circ ,$$

hence,

$$(1/3)\,\text{ext. } \angle MCN = 120^\circ - ((1/3)\,\text{ext. } \angle ACD - 60^\circ) ,$$

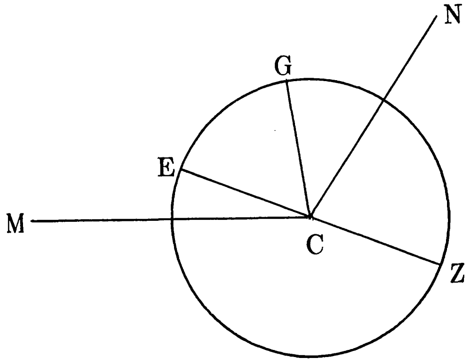
$$(1/3)\,\text{ext. } \angle MCN = 180^\circ - (1/3)\,\text{ext. } \angle ACD .$$

But ∠ ACZ is (1/3) ext. ∠ ACD and 180° − ∠ ACZ is ∠ ACG. Thus, ∠ ACG is (1/3) ext. ∠ MCN.

IV. To trisect any angle over 90° and under 180°.

Problem: Trisect ∠ MCN.

Construction: Construct circle with centre at C and any radius. Construct Z in circle so that ∠ ZCN is (1/3) ext. ∠ MCN. Let line ZC meet circle at E. Construct G on circle between E and N so that EG is one



radius. Then $\angle NCG$ is $(1/3)\angle MCN$.

Proof: ΔEGC is isosceles and so $\angle ECG$ is 60°. But

$$(1/3)\angle MCN \text{ is } 120° - (1/3) \text{ ext. } \angle MCN$$
$$\text{is } 120° - \angle ZCN .$$

But $\angle ZCE$ is 180°, $\angle ECG$ is 60° and so $\angle ZCG$ is 120°. Hence

$$(1/3)\angle MCN \text{ is } \angle ZCG - \angle ZCN ,$$

i.e.

$$(1/3)\angle MCN \text{ is } \angle NCG .$$

Binary Expression of Arabic Baroque

0

The circle, whole.
The circle, naught.
All naught.

1

Silence, darkness, nothing.
Sound, light, the line.
I one. One alone.

1 + 1

One and another,
1 *more than* 1, *contained*
in the next place.

10

Naught all.
I move empowered
to infinity.

Devera Sievers

A CONSTRUCTION BY RULER ONLY

C. N. BHASKARANANDHA, Mercara, Coorg, India

We refer the reader to *A course of pure geometry* by E. H. Askwith, D. D. wherein at the end of the chapter on cross ratios, a solution is provided for the following problem: "construct a straight line parallel to a given pair of parallel straight lines through a given point, the construction being effected by the use of an ungraduated ruler only." Reference may be made to page No. 55, and article No. 57, in the Indian Edition published by K. K. Bhaghat in the year 1955.

The following method of construction which is more elementary and less complicated is of interest.

Let "l" and "m" be the given pair of parallel straight lines and P, the given point in their plane.

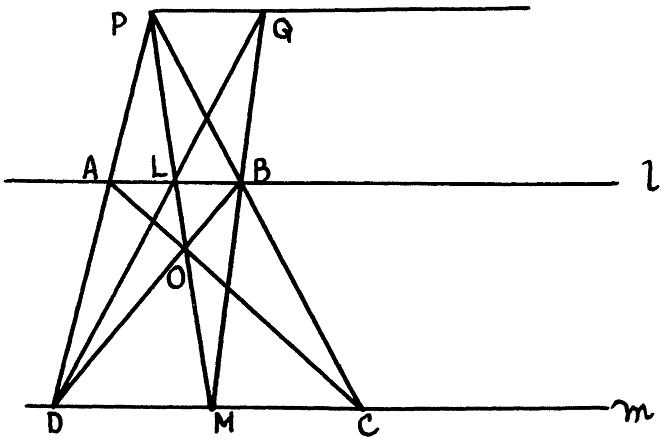


FIG. 1

Construction: Take any two straight lines through P cutting "l" in A and B and "m" in C and D (fig. 1). Join AC and BD. Let AC and BD intersect in O. Let PO cut AB in L and CD in M. Join DL and MB and produce to meet in Q. We shall prove that PQ, "l" and "m" are mutually parallel.

Proof. We may make use of the harmonic properties of the quadrilateral comprised of the vertices P, B, C, O, D and A and remark that

$$AL = LB \quad \text{and} \quad DM = MC \; .$$

To make the proof more elementary we shall restrict ourselves to the use of the properties of similar triangles only. The triangles PAL and PLB are respectively similar to the triangles PDM and PMC. So

$$\text{(X)} \qquad \frac{AL}{DM} = \frac{PL}{PM} = \frac{LB}{MC} \; .$$

Also the triangles ALO and LBO are respectively similar to triangles MCO and MDO. Hence

(Y) $$\frac{AL}{MC} = \frac{LO}{OM} = \frac{LB}{DM} .$$

From X and Y it follows that

$$\frac{AL}{LB} = \frac{DM}{MC} = \frac{MC}{DM} .$$

Whence

$$AL = LB \quad \text{and} \quad DM = MC .$$

Now triangle QLB is similar to triangle QDM so,

$$\frac{LB}{DM} = \frac{QB}{QM} ; \quad \text{but} \quad DM = MC$$

therefore

(Z) $$\frac{LB}{MC} = \frac{QB}{QM} .$$

From X and Z,

$$\frac{QB}{QM} = \frac{PL}{PM}$$

therefore PQ is parallel to the pair "l" and "m".

Note: Fig. 1 represents the case when P is outside "l" and "m". For the case when P is between "l" and "m", the following figure (fig. 2) holds with the same proof.

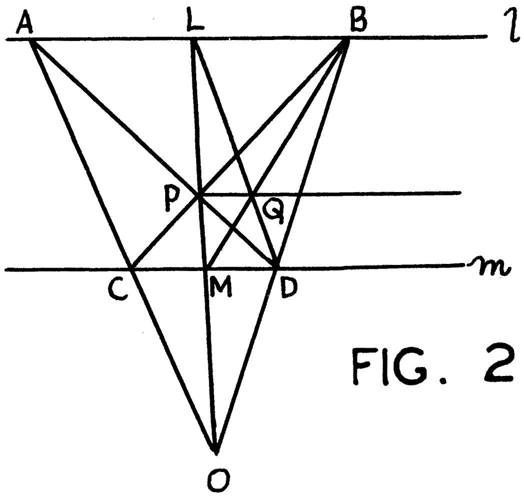


FIG. 2

TEACHING OF MATHEMATICS

Edited by ROTHWELL STEPHENS, Knox College

This department is devoted to the teaching of mathematics. Thus, articles of methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher,* are interested, or questions which you would like others to discuss, should be sent to *Rothwell Stephens, Mathematics Department, Knox College, Galesburg, Illinois.*

A USE OF INEQUALITIES FOR LOCI IN ANALYTIC GEOMETRY

ROBERT I. JENNRICH and RAYMOND B. KILLGROVE,*

University of California, Los Angeles

In classical geometry a locus is basically a set of points (or lines) with a property; e. g. "the circle is the locus of points equidistant from a given point called the center" could be restated as "the set C of points p of a two-dimensional Euclidean space X where $C = \{p \mid d(p, a) = k\}$, a is a fixed point, d is a distance function, and k is a nonnegative real number. In this case we have a definition, hence there is nothing to prove. On the other hand, the statement "the locus of points equidistant from two given points is the perpendicular bisector of the line segment joining these points" has to be proved. In the proof one shows (1) every point which satisfies the locus property lies on the perpendicular bisector, and (2) every point which lies on the perpendicular bisector satisfies the locus property.

In a majority of texts on analytical geometry, the proof including both the necessary and sufficient conditions of a locus problem is not made. Whether or not the complete proof should be made for each locus problem is not the concern of this note. Instead it is suggested that it could be done for the conic sections, or at least for the ellipse. The discussion which follows describes the proof for the ellipse only, but this method should be applicable to other locus problems as well.

Let us proceed as follows: we define the ellipse to be the locus of points where the sum of the distances from a point on the locus to each of two fixed points called foci is a constant distance (greater than the distance between the foci). In the coordinate plane, let the constant sum of the distances be $2a$, and the foci be $(c, 0)$ and $(-c, 0)$, $a > c > 0$. Now proceed in one of the several obvious ways and obtain the result that every point (x, y) satisfying the locus property, satisfies the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 . \tag{1}$$

*The preparation of this paper was sponsored by the Office of Naval Research.

To complete the proof we want to show that every point (x, y) satisfying equation (1) has the locus property, namely the sum of the distance from (x, y) to $(c, 0)$ and $(-c, 0)$ is $2a$. Since the steps are not reversable without some additional arguments to eliminate other cases which appear, we suggest the following alternate procedure. Suppose there is a point P_1 with coordinates (x_1, y_1) which satisfies the equation (1) but not the locus property. There is a distance defined between P_1 and each of the foci, and let the sum of these distances be $2k$. We note by the triangular inequality or shortest distance arguments that $k \geqq c$, and in fact $k = c$ implies that the point (x_1, y_1) lies on the segment joining the foci. Then in this latter case, $y_1 = 0$, and since the point satisfies the equation, $x_1 = \pm a$. These two points $(a, 0)$ and $(-a, 0)$ do satisfy the locus property trivially, hence $k > c$.

Now we know from the previous work that (x_1, y_1) also satisfies the equation

$$\frac{x^2}{k^2} + \frac{y^2}{k^2 - c^2} = 1 . \tag{2}$$

Since $k \neq a$, then $k < a$ or $k > a$. Suppose $k < a$, then $k^2 < a^2$.

$$k^2 - c^2 < a^2 - c^2 , \quad \frac{1}{k^2} > \frac{1}{a^2} , \quad \frac{1}{k^2 - c^2} > \frac{1}{a^2 - c^2} ,$$

$$\frac{x_1^2}{k^2} > \frac{x_1^2}{a^2} , \quad \frac{y_1^2}{k^2 - c^2} > \frac{y_1^2}{a^2 - c^2} ,$$

$$\frac{x_1^2}{k^2} + \frac{y_1^2}{k^2 - c^2} > \frac{x_1^2}{a^2} + \frac{y_1^2}{a^2 - c^2} ,$$

contradicting the fact that the point P_1 could satisfy both equations (1) and (2) One can repeat these steps for $k > a$.

The use of this proof gives one an opportunity to use inequalities and the indirect method of proof in analytic geometry. The student should have seen both in a course in classical geometry. He may have seen the former in courses in algebra either in the secondary school or in college as well.

MISCELLANEOUS NOTES

Edited by ROY DUBISCH, University of Washington

Articles intended for this department should be sent to Roy Dubisch, Department of Mathematics, University of Washington, Seattle, Washington.

AN APPROXIMATION FOR ANY POSITIVE INTEGRAL ROOT

GERALD D. TAYLOR, San Jose State College

It was brought to the attention of a colleague of mine, Professor Hoggatt, that one was in need of a method to get a cube root on a hand calculator which took only square roots. He suggested the following procedure: To obtain $\sqrt[3]{A}$ first approximate $\sqrt[3]{A}$ by A_0. Then use the sequence $A_{n+1} = \frac{1}{2}(A_n + \sqrt{A/A_n})$, $n = 0, 1, 2, \cdots$, to improve this approximation up to the limits of accuracy of the calculator.

The generalization I propose is described in the following theorem.

Theorem: If $L = \sqrt[s]{A}$ for $A > 1$, $s = 2^k - h$ where $1 \leqq h \leqq 2^k - 1$ (h and k are positive integers), and if A_0 is any number such that $A > A_0^s$, then $L = \lim_{n\to\infty} A_n$, where

$$A_{n+1} = \gamma A_n + \beta \sqrt[s+h]{A_n^h A}\,, \quad n = 0, 1, 2, \cdots, \quad \text{and} \quad \gamma + \beta = 1\,, \quad 0 \leqq \beta, \gamma\,.$$

Proof: To show that this recursion sequence converges, we show that it is bounded and monotone increasing. First we show that it is bounded as follows:

$$A_1 = \gamma A_0 + \beta \sqrt[s+h]{A_0^h A} < \gamma A^{1/s} + \beta \sqrt[s+h]{A^{h/s} A} = A^{1/s}(\gamma + \beta) = A^{1/s}\,,$$

since both γ and β are nonnegative. Thus $A > A_0^s$ implies $A > A_1^s$. Now assume that $A > A_n^s$, then

$$A_{n+1} = \gamma A_n + \beta \sqrt[s+h]{A_n^h A} < \gamma A^{1/s} + \beta \sqrt[s+h]{A^{h/s} A} = A^{1/s}(\gamma + \beta) = A^{1/s}\,.$$

Hence by induction we have that $A > A_n^s$ for all n and thus the sequence is bounded. Now to show that the sequence is strictly monotone increasing we note that

$$\frac{A_{n+1}}{A_n} = \frac{\gamma A_n + \beta \sqrt[s+h]{A_n^h A}}{A_n} = \gamma + \beta \sqrt[s+h]{A_n^{-s} A} > \gamma + \beta \sqrt[s+h]{A_n^{-s} A_n^s} = \gamma + \beta = 1\,,$$

since $A > A_n^s > 1$ for all n by the previous argument.

Thus since the sequence is both bounded and strictly monotone increasing, it has a limit L'. Since

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} A_{n+1} = L',$$

we have that

$$L' = \gamma L + \beta \sqrt[s+h]{L^h A},$$

since fractional powers are continuous for positive arguments. Thus,

$$L'(1-\gamma) = \beta \sqrt[s+h]{(L')^h A}$$

or, since $1-\gamma = \beta$,

$$(L')^{s+h} = (L')^h A \quad \text{and} \quad L' = \sqrt[s]{A} = L .$$

For the case when $A < 1$, we need only factor out an appropriate negative power of ten and proceed as above. Also, if $s = 2^k$ we may obtain L directly by repeated square root extraction.

The β and γ present in the theorem could be thought of as convergence factors, as proper choice of them can speed up the rate of convergence of the sequence. The rate of convergence will be the greatest with $\gamma = 0$ and $\beta = 1$, and it will decrease as one increases γ and decreases β from these values. Thus we end the article with the illustration of the seventh root of 128, which is 2, with two different choices of β and γ in the following table.

TABLE

	Choice 1	Choice 2
	$\beta = \frac{1}{2}$, $\gamma = \frac{1}{2}$	$\gamma = 0$, $\beta = 1$
A_0	1	1
A_1	1.417004043	1.834008085
A_2	1.666341288	1.978456025
A_3	1.810614328	1.997294225
A_4	1.892949037	1.999661584
A_5	1.939621703	1.999957694
A_6	1.965986455	1.999994711
A_7	1.980851393	1.999999338
A_8	1.989223865	1.999999916

COMMENTS ON PAPERS AND BOOKS

Edited by HOLBROOK M. MACNEILLE, Case Institute of Technology

This department will present comments on papers published in the MATHEMATICS MAGAZINE, lists of new books, and reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent to *Holbrook M. MacNeille, Department of Mathematics, Case Institute of Technology, Cleveland 6, Ohio.*

NOTE ON "THE EXTENSION OF PASCAL'S THEOREM"

C. E. MALEY, The Carborundum Company

Several readers have stated that they haven't "got to first base" in extending Pascal's Theorem (in metric form) to 3-space and have asked if I had any particular approach in mind that will get them "on the right track".

As the writer has not yet made the extension himself, he is in position to do only just this.

Notice that

$$\phi(x^2, x, 1)_1 = 2\begin{vmatrix} M_1^x & L_1 \\ M_2^x & L_2 \end{vmatrix} = 2L_1L_2L\ ,$$

$$\phi(x^2, xy, y^2, x, y, 1)_1 = 8\begin{vmatrix} M_1^x & M_1^y & A_1 \\ M_2^x & M_2^y & A_2 \\ M_3^x & M_3^y & A_3 \end{vmatrix} = 16A_1A_2A_3A\ ,$$

where the L's, A's and M's are lengths, areas and "moments," respectively.

Also, replace "triangle" by "tetrahedron."

BOOK REVIEWS

More Numbers: Fun and Facts. By J. Newton Friend. Charles Scribner's Sons, New York, 1961, xiv + 201 pp. hard cover, \$2.95.

For years before his retirement from the Chemistry Department at the Technical College of Birmingham, England, Dr. Friend pursued his hobby of collecting odd and amusing facts about numbers. In 1954 he published some of his collection in the 208-page *Numbers: Fun and Facts.* The current volume follows the general pattern of the first one, with some slight repetition but with more careful authentication of the "facts." However, the hodge-podge of problem and fact entitled "Chapter I Noah's Ark" does include the statement "Five of the nine digits are odd and include all four of the digit primes."

"The Digits in Sequence" deals with relationships involving distinct digits and considerable folk-verse about numbers. "Prime Numbers" are disposed of with some curiosa and miscellaneous problems. "Problems with Digits and Letters" points up some numerical relations necessary for solution of cryptarithms and offers a nice selection of problems. An elementary introduction to "Squares, Magic and Otherwise" is provided. Many fascinating oddities are collected in "The Digits and Superstition."

The last half of the book is devoted to 100 "Miscellaneous Problems." As might be expected, the verbiage and content, particularly of the monetary problems, have the flavor of the British Isles. The problems are aimed at the "Man in the Street" and are appropriately chosen. Answers are given in Appendix II.

The typography is very good. The text is written in a flowing style. Non-precise definitions are usually clarified by numerical examples. Multiple solutions are sometimes designated "unique." Solecisms such as "to try and arrange" are few. Appendix I contains primes, squares and cubes to 100 and primes between 100 and 1009. There is an adequate index but no bibliography.

Mathematical hobbyists and high-school students should find this book interesting and stimulating. It should find a place on the browsing shelf of every school and public library.

Charles W. Trigg, Los Angeles City College

Foundations of modern mathematics. By Jean Dieudonne. Academic Press, Inc., New York, 1960, 354 pp.

This text, intended as a first year graduate course in analysis, is an admirable introduction to modern analysis distinguished by precision of language, adherence to the axiomatic method and a presentation of fundamental concepts of analysis within the broad framework of metric and Banach spaces.

Chapter I presents an irreducible minimum of the basic material concerning relations, functions and mappings. The real number system (Chapter II) is introduced as an archimedian ordered field satisfying the nested interval property. Next (Chapter III) comes the basic material on metric spaces including sections on locally compact, connected and product spaces, but, oddly enough, excluding the Baire category theorem. Chapter IV, a return to the real line, begins with a study of continuity of algebraic operations from $R \times R$ into R which includes, for example, a proof that all intervals in R are homeomorphic to R. Following a brief discussion of monotone functions the logarithm of base a is introduced as follows: "For any number $a > 1$, there is a unique increasing mapping f of $R_+^* =]0, +\infty[$ into R such that $f(xy) = f(x) + f(y)$ and $f(a) = 1$; moreover f is a homeomorphism of R_+^* onto R." The existence of such a function is established by means of an adroit application of the least upper bound principle. The chapter closes with a proof of the Tietze-Urysohn extension theorem for metric spaces in which the formula for the extending function is given explicitly. Chapter V sets forth the basic facts on normed and Banach spaces with special attention given to continuity conditions and builds up to the F. Riesz theorem characterizing finite dimensional spaces. After a short chapter on Hilbert spaces there somes Chapter VII on spaces of continuous functions. The Stone-Weierstrass theorem is presented along with the classical Arzela-Ascoli theorems on compactness of families of continuous functions. By way of preparing for contour integration, the author introduces the class of regulated functions (piece-wise continuous functions), those functions from a real interval I into a Banach space which have left and right limits in the norm topology of the Banach space at each point of I. It is shown that such functions are uniform limits of step-functions. Chapter VII treats differential calculus from the point of view of *local* approximation of functions by linear forms. "Let E, F be Banach spaces, A an open set in F, f, g mappings from A into F; we say that f and g are tangent at $x_0 \in A$ if $||f(x) - g(x)|| \cdot ||x - x_0||^{-1} \to 0$ as $x \to x_0$." It is established that among all functions g tangent to f at x_0, at most one is of the form $x \to f(x_0) + u(x - x_0)$ where u is linear. When this mapping u exists it is called the *derivative* of f at the point x_0, and written $f'(x_0)$. The derivative (when it exists) of a continuous mapping f of A into F, at a point $x_0 \in A$, is an element of $\mathsf{L}(E, F)$, the continuous linear maps from E into F. From this definition the usual formal differential rules are derived which, of course, reduce to the classical ones when E and F are taken to be the real line. Various applications of a mean-value inequality for vector-valued functions are given. Upon this notion of derivative and the class of regulated functions, the author builds a simple integration theory sufficient for later applications but requiring no techniques of measure theory. The remainder of the chapter is devoted to a study of Jacobians, higher derivatives and Taylor's formula. Following the next chapter, a treatment of analytic functions which employs functions of several variables wherever practicable and presents a proof of Cauchy's theorem utilizing the concept of homotopic paths, is an appendix devoted to a study of certain deep topological properties of the plane through the techniques of complex integration. In Chapter X the methods of successive approximation are applied to implicit functions, the rank theorem and the Cauchy existence theorems for ordinary differential equations. The last chapter deals with elementary spectral theory for compact operators with applications to the Fredholm theory and Sturm-Liouville problems.

Finally, "something really different" – fairly describes, in the opinion of the reviewer, this text which may well be the best in its class.

James Wells, University of North Carolina

Geometric Inequalities. By Nicholas D. Kazarinoff. Random House, Inc., New York, 1961, paper back, 132 pp., $1.95.

This book is the fourth volume of the New Mathematical Library series sponsored by the School Mathematics Study Group. Its aim, like that of the five others thus far published, is to present some important mathematical ideas in a way that highschool students and laymen would find attractive and understandable. Since this commendable objective has generally been notoriously difficult to achieve, it is impossible to predict how successful this attempt will be with the audience toward which it is directed. On the other hand, more advanced students will undoubtedly be pleased with many of the assorted items of interest that appear, and it is very likely that this book will serve as a popular source for the kind of enrichment material so eagerly sought by our more dedicated high-school and college teachers.

The author presents his subject in an easy-going, conversational style and gently guides the reader into the deeper intricacies of geometric inequalities. He combines informality with mathematical rigor and displays a brand of praiseworthy enthusiasm that is bound to rub off on the reader. In many cases he offers several different proofs of the same theorem, thus encouraging an appreciation for the neat and esthetic demonstration.

The book makes no pretense of attempting a complete treatment of a practically inexhaustible subject. Chapter One deals with the fundamentals of arithmetic and geometric means and serves as more than a foundation for notions elaborated in the main body of the work. Indeed, from the standpoint of unity and consistency, much of the material is irrelevant to the main theme of geometric inequalities. Still, the subject is interesting for its own sake and the reader is urged to refer to the companion volume on Inequalities, by Beckenbach and Bellman, for more detailed study. The second chapter considers a number of selected classical isoperimetric theorems and is liberally sprinkled with a variety of proofs and an assortment of provocative problems, some solved and others unsolved. Chapter Three expounds the Reflection Principle and gives examples of the application of symmetry in the solution of geometric problems. The ideas developed in this section culminate in a detailed exposition of the proof of the Erdos-Mordell Theorem devised by the author's father, Donat K. Kazarinoff. The final chapter is devoted to hints and solutions for almost all the problems proposed.

Since the work is intended to stimulate the neophyte and perhaps to direct him along the path to further study, it is regrettable that an index and a formal bibliography are lacking, although some historical material and miscellaneous references are scattered throughout the text. Whetting the appetite of the novice is a laudable undertaking, but satisfying it is even more admirable. This book could have served its purpose far better if it had included more suggestions for collateral reading. For example, no mention is made of the excellent proof of the Erdos-Mordell Theorem published by H. G. Eggleston in the Mathematical Gazette, vol. 42, 1958, pp. 54-55. The inspired and ambitious student might perhaps have been interested in the pertinent material contained in an article by Victor Thebault, appearing as a supplement to Mathesis, December 1958. Still, some omissions are forgivable. It is likely that this book was already in press when Mordell discovered his improved proof of the Erdos-Mordell Theorem, described in the answer book accompanying H. S. M. Coxeter's *Introduction to Geometry.*

Some of the proofs given in the text appear to be far more intricate and belabored than those given in, for example, McClelland's *Geometry of the Circle* or in Rademacher and Toeplitz's *The Enjoyment of Mathematics.* We can explain this by assuming that it was probably the intention of the author to encourage cerebral participation on the part of the reader.

One can hardly quarrel with the author for desiring his book to be a tribute and a memorial to the memory of his late father, who taught and inspired him. However, the statement on page 116 that D. K. Kazarinoff had a proof of the general result for the three-dimensional analog of the Erdos-Mordell Theorem but refused to divulge it does not go over well with mathematicians in this day and age. Ever since the run-around created by Fermat's unsubstantiated marginal note, mathematicians have been careful to consider a proof a proof only after it has been published, revealed and disseminated. Secrets of this sort have no place in the art and science of mathematics.

Notwithstanding its few, minor shortcomings, the book is interesting and stimulating and serves as a reminder that Geometry still lives in this Century of Analysis.

Leon Bankoff, Los Angeles, California

Nabla, The Bulletin of the Malayan Mathematical Society. Quarterly. Edited by P. Lancaster, Department of Mathematics, University of Malaya, Singapore, 10. $2.00 (Malayan) per issue, free to members of the Society, membership $10.00 per annum.

This interesting, attractive, $6\frac{1}{2}'' \times 8''$ quarterly averages about 40 pages per issue. It is printed by the Gestelith process by Gestetner (Eastern) Ltd., Singapore, and is partially subsidized by the *Asia Foundation.* The composition is good, the type readable, and the figures well drawn.

NABLA "is intended to help in bridging the various gaps between professional and amateur mathematicians, students and teachers of mathematics. The level of mathematical abstraction aimed at is somewhere near university entrance level, although fairly wide variations are permitted on either side of this."

Some of the articles are nicely selected reproductions from other magazines, some are reports of lectures before the Society, and others appear there for the first time. Typical titles are:

A. Oppenheim. *Topics in Number Theory.*

Dan Pedoe. *Non-Euclidean Geometry.*

R. K. Guy. *Some Mathematical Recreations.*

P. Lancaster. *Approaching the Speed of Sound.*

A. C. Aitken. *The Art of Mental Calculation.*

G. P. Darke. *The Use of Meccano in Elementary Mathematics Teaching.*

E. P. Wigner. *The Unreasonable Effectiveness of Mathematics in the Natural Sciences.*

R. A. Rankin. *On Sequences of Integers Containing No Arithmetical Progressions.*

M. J. Wicks. *The Diophantine Equation* $4abc = (a+b+c)(a+b-c)(a-b+c)$.

What otherwise would be blank space at the end of the articles is filled with "JOTTINGS", which are well-chosen quotations relating to mathematical concepts from a wide variety of authors and publications, old and new.

The "Notes" section, they are up to N. 118 by now, contains short articles and comments on papers and problems previously published.

The Problems section edited by R. K. Guy (soon to be replaced by M. J. Wicks) consists of Solutions to Earlier Problems, followed by from 6 to 11 Further Problems submitted by readers. These problems may be original or may have appeared previously in various competitions, books, or other magazines.

NABLA would make an instructive addition to the magazine shelves of American junior colleges, four-year colleges and universities.

Charles W. Trigg, Los Angeles City College.

BOOKS RECEIVED FOR REVIEW

The Theory of Determinants, Matrices, and Invariants. By H. W. Turnbull. Dover, New York, 1961, xviii + 370 pp., $2.00.

Finite Groups. By G. A. Miller, H. F. Blichfeld, and L. E. Dickson. Dover, New York, 1961, xvii + 381 pp., $2.00.

Elliptic Functions. By F. Fowman. Dover, New York, 1961, 115 pp., $1.35.

Theory of Functions. By R. Rothe, F. Ollendorff, and K. Pohlhausen. (Translated by A. Herzenberg). Dover, New York, 1961, x + 189 pp., $1.35.

Summation of Series. By L. B. W. Jolley. Dover, New York, 1961, xii + 251 pp., $2.00.

Lectures on the Calculus of Variations. By Oskar Bolza. Dover, New

York, 1961, xi + 271 pp., $1.65.

Tables of Indefinite Integrals. By G. Petit Bois. Dover, New York, 1961, 149 pp., $1.65.

Fallacies in Mathematics. By E. A. Maxwell. Cambridge University Press, 1959, 95 pp., $2.95.

TO SPACE

In thee we are, and live, and move.
Thou art our frame of reference,
And it would hardly well behoove
Thy guests to risk the charge that you've
Not had the proper deference.

The axioms of thy chrysalis
We recast, quite indecorous;
Recasting even, worse than this,
Aspersions on thy genesis.
And where, pray tell, does this leave us?

MARLOW SHOLANDER

PROBLEMS AND SOLUTIONS

Edited by ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California.*

PROPOSALS

474. *Proposed by Murray S. Klamkin, AVCO, Wilmington, Massachusetts.*

The three polynomials $x-x$, x^2+y^2-2xy, and $x^3+y^3+z^3-3xyz$ can each be factored into real polynomials. Which if any of the higher order analogous polynomials

$$\sum_{r=1}^{n} x_r^n - nx_1x_2\cdots x_n$$

are reducible?

475. *Proposed by Guy G. Becknell, Tampa, Florida.*

Find a unique set of six integers which form a group having the peculiar property that the product of any five of them is one or more periods of the unit repetend number of the remaining one. For example

$$\frac{1}{41} = .\overline{02439}\ .$$

Hence, the unit repetend number for 41 is 02439, equivalent to 2439 numerically.

476. *Proposed by Kaidy Tan, Fukien Normal College, China.*

Draw a straight line bisecting the perimeter and area of a given quadrilateral.

477. *Proposed by Maxey Brooke, Sweeney, Texas.*

Before they began, Brown had twice as much money as Calhoun while Adams had three times as much. The game was called Sudden Death. Two cards are dealt to each player. He puts into the pot an amount equal to the product of those cards (jacks, queens, and kings count as 11, 12, and 13). A third card is dealt and the one receiving the low third card wins the pot. During the evening there were losses and wins. But on the last hand, Adams put one-half the money he had in the pot, Brown put in one-third of his money, and Calhoun put in one-sixth. Each player drew a three as the last card and they split the pot equally. On counting up, each player found that he had the same amount that he began with. How much was it?

478. *Proposed by C. W. Trigg, Los Angeles City College.*

a) Identify the four five-digit palindromic numbers whose squares are composed of distinct digits.

b) The square of a permutation of one of these palindromes also has distinct digits. Find it and show that it is the only one.

479. *Proposed by M. N. Gopalan, Mysore City, India.*

ABC is a right triangle with right angle at C. CD is drawn perpendicular to AB. r_1 and r_2 are the in-radii of the triangles formed and r is the in-radius of triangle ABC. Prove that $r_1^2 + r_2^2 = r^2$.

480. *Proposed by Gilbert Labelle, College de Longueuil, Canada.*

Prove that $2^p - 1$ is not a prime number if the prime p is of the form $4m-1$ and $2p+1$ is also a prime number.

SOLUTIONS

LATE SOLUTIONS

428. *Leon Bankoff, Los Angeles, California.*
446. *A. J. Kokar, Adelaide, Australia.*
447, 448. *A. J. Kokar, Adelaide, Australia* and *C. W. Trigg, Los Angeles City College.*

COMMENTS ON SOLUTIONS

428. [November 1960 and May 1961] *Comment by C. F. Pinzka, University of Cincinnati.*

The numbers L and M in Solution II do not quite solve the problem, inasmuch as kL and kM will have more digits than L and M for $k > 2$ and $k > 1$, respectively. One way of remedying this situation is to write $L = 034\cdots931$ and $M = 058\cdots647$. However, if initial digits of zero are not allowed, then we must search for cycles M such that $2M$, $3M$, $4M$, $\cdots$, $7M$ contain the same number of digits as M. Clearly, this restricts the first three digits of M to 100, 101, $\cdots$, 142.

The smallest denominator which yields a period greater than 6 is 17, and we find that the digits in one period of 1/17 are 0 58823 52941 17647. Noting that a cycle of maximum length has the digit-preserving property if the multiplication does not increase the number of digits, and that this property holds for cyclic permutations of such a cycle, we find that $M = 1\ 17647\ 05882\ 35294$ has the digit-preserving property for $k \leqq 8$.

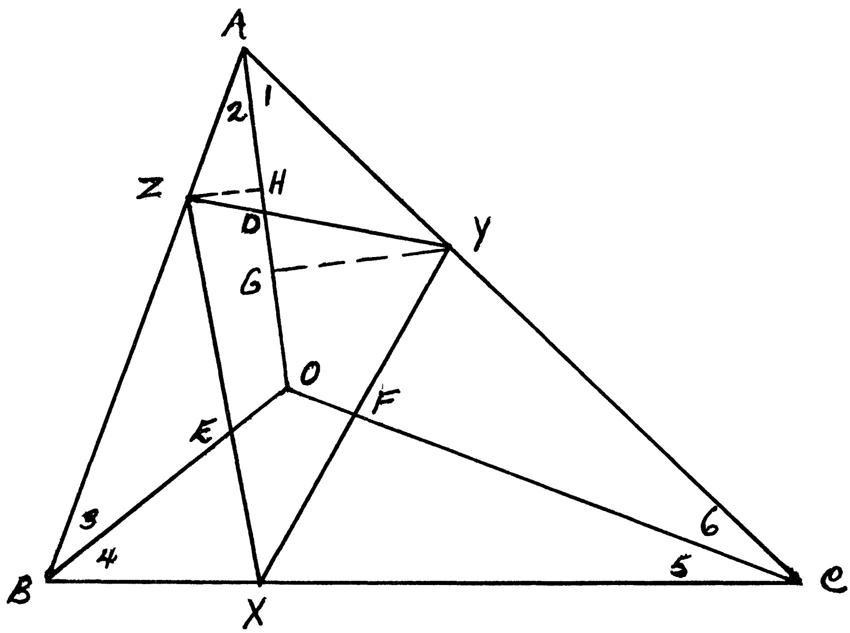
Examining other denominators, we find the following:

Denominator	*M*	*k*
19	105 26315 78947 36842	$\leqq 9$
23	13 04347 82608 69565 21739	$\leqq 7$
29	137 93103 44827 58620 68965 51724	$\leqq 7$
29	103 44827 58620 68965 51724 13793	$\leqq 9$

432. [January 1961 and September 1961] *Comment by D. Moody Bailey, Princeton, West Virginia.*

It may be noted that the desired result holds when triangle XYZ is any triangle whose vertices lie on the sides of triangle ABC (It is not necessary that XYZ be the pedal triangle or even a Miquel triangle of point O with respect to triangle ABC). The reasoning runs as follows:

Allow XYZ to be any triangle whose vertices X, Y, Z lie on sides BC, CA, AB of triangle ABC. Let O be any point in the plane of triangle ABC and construct AO, BO, CO to meet sides YZ, ZX, XY at respective points D, E, F. Allow the angles at the vertices of triangle ABC to be numbered as shown. From Y and Z drop perpendiculars to meet AO at



respective points G and H. Right triangles DZH and DYG are then similar and $\frac{ZD}{DY} = \frac{ZH}{YG}$. Considering right triangles AZH and AYG, we obtain

$$\frac{ZH}{YG} = \frac{AZ}{YA} \cdot \frac{\sin 2}{\sin 1}.$$

Substitution then yields

$$\frac{ZD}{DY} = \frac{AZ}{YA} \cdot \frac{\sin 2}{\sin 1}.$$

In similar fashion

$$\frac{YF}{FX} = \frac{CY}{XC} \cdot \frac{\sin 6}{\sin 5} \quad \text{and} \quad \frac{XE}{EZ} = \frac{BX}{ZB} \cdot \frac{\sin 4}{\sin 3}.$$

Consequently,

$$\frac{ZD}{DY}\cdot\frac{YF}{FX}\cdot\frac{XE}{EZ}=\frac{AZ}{YA}\cdot\frac{CY}{XC}\cdot\frac{BX}{ZB}\cdot\left(\frac{\sin 2}{\sin 1}\cdot\frac{\sin 6}{\sin 5}\cdot\frac{\sin 4}{\sin 3}\right).$$

As AO, BO, CO are concurrent, the expression in parenthesis is equal to unity. Accordingly, we may re-arrange the right member of the preceding equation and have

$$\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=\frac{ZD}{DY}\cdot\frac{YF}{FX}\cdot\frac{XE}{EZ}.$$

Hence the equality is true for all positions of point O in the plane of triangle ABC and for any triangle XYZ whose vertices lie on the sides of triangle ABC.

Polynomial Roots

453. [September 1961] *Proposed by Joseph W. Andrushkiw, Seton Hall University.*

a) If a root of the polynomial $f(x)$, whose roots are real, is of multiplicity three or greater, show that $F(x) = f(x)+c$, $c \neq 0$, cannot have all roots real.

b) The polynomials $f(x)$ and $g(x)$ have all real roots and $f(x) = g(x)+c$, $c > 0$. Prove that $h(x) = g(x)+k$, $0 < k < c$, has all roots real and distinct.

Solution by C. W. Trigg, Los Angeles City College.

a) If $f(x) = 0$, of degree n with all roots real, has a root of multiplicity $m \geq 3$, the graph of $y = f(x)$ has m coincident zeros at its point of tangency with the x-axis. Any line parallel to the x-axis and close to it will intersect the curve near the point in one place if n is odd, or in two or no places if n is even. Thus in the transformation $F(x) = f(x)+c$, $c \neq 0$, at least two real roots must be lost.

b) If $g(x) = 0$ has all n of its roots real, then the graph $y = g(x)$ has n real zeros. If the line $y = -c$ cuts the graph in n points (where tangencies are counted as two points each), then $y = g(x)+c$ will have all real roots. The x-axis and the line $y = -c$ cut or have two-point tangencies to all loops of the graph, so any line $y = -k$, $0 < k < c$, between them will cut the graph in n distinct points. Hence, $h(x) = g(x)+k = 0$ will have all of its roots real and different.

Also solved by P. D. Goodstein, University of Leicester, England; Gilbert Labelle, Collège de Longueuil, Canada; Richard Levitt, Boston Latin School; and the proposer.

An Angle Bisector

454. [September 1961] *Proposed by C. N. Mills, Sioux Falls College, South Dakota.*

Given the sides a and b and the included angle $c = 2\theta$ of a triangle. Prove that the length of the bisector of angle c is equal to $(2ab\cos\theta)/(a+b)$.

I. *Solution by Gilbert Labelle, Collège de Longueuil, Canada.* Draw two oblique cartesian axes OX and OY with included angle 2θ. Let $OA = b$ and $OB = a$. The bisector of the angle 2θ is represented by the equation $x = y$ and the third side AB of the triangle is given by the equation

$$\frac{x}{b} + \frac{y}{a} = 1 .$$

Solving these equations we have

$$x_1 = y_1 = \frac{ab}{a+b} .$$

The length of the bisector is given by the law of cosines

$$(OM)^2 = x_1^2 + y_1^2 + 2x_1y_1 \cos 2\theta = \left(\frac{ab}{a+b}\right)^2 \cdot (2 + 2\cos 2\theta)$$

or

$$OM = \frac{ab}{a+b}\sqrt{2 + 2(2\cos^2\theta - 1)} = \frac{2ab\cos\theta}{a+b} .$$

II. *Solution by Joseph B. Bohac, St. Louis, Missouri.* The angle bisector divides the triangle into two triangles with common point C and angles θ. If we call the length of the bisector L and note that the sum of the areas of the smaller triangles must equal that of the larger, we have

$$\tfrac{1}{2} aL \sin\theta + \tfrac{1}{2} bL \sin\theta = \tfrac{1}{2} ab \sin 2\theta$$

or

$$(a+b)L \sin\theta = ab \sin 2\theta = 2ab \sin\theta\cos\theta .$$

Therefore

$$L = \frac{2ab\cos\theta}{a+b} .$$

III. *Solution by Sister M. Stephanie, Georgian Court College, New Jersey.* By Stewart's theorem, with t_c the bisector of angle C:

$$(1) \qquad t_c^2 = ab\left[1 - \frac{c^2}{(a+b)^2}\right] .$$

Using the law of cosines for this triangle : $c^2 = a^2 + b^2 - 2ab\cos 2\theta$ and the identity $\cos 2\theta = 2\cos^2\theta - 1$ and substituting in (1) we get

$$t_c = \frac{2ab\cos\theta}{a+b} .$$

IV. *Solution by M. Morduchow, Polytechnic Institute of Brooklyn.* Let L, x, y be, respectively, the length of the angle bisector and the segments into which the third side is divided by it. Then

$$(1) \qquad \frac{x}{y} = \frac{a}{b}$$

(2) $$x^2 = a^2 + L^2 - 2aL\cos\theta$$

(3) $$(x+y)^2 = a^2 + b^2 - 2ab\cos 2\theta \,.$$

Solving (1) for y in terms of x, substituting into (3) and then replacing x^2 there by the right side of (2), one obtains :

$$L^2 - 2aL\cos\theta + \frac{2a^3b}{(a+b)^2}(1+\cos 2\theta) = 0 \,.$$

Since $1+\cos 2\theta = 2\cos^2\theta$, the roots of the above equation are

$$L = \frac{2ab\cos\theta}{a+b} \text{ and } \frac{2a^2\cos\theta}{a+b} \,.$$

The latter is rejected, since L must be a symmetric function of a and b here.

V. *Solution by Roger D. H. Jones, College of William and Mary, Virginia.* Let CF be the bisector of Angle C: let AD and BE be drawn perpendicular to CF. Then

$$\text{Area of triangle } CAB = \tfrac{1}{2}\cdot CF\cdot(AD+BE) = \tfrac{1}{2}\cdot CF\cdot(b\cdot\sin\theta + a\cdot\sin\theta) \,.$$

But

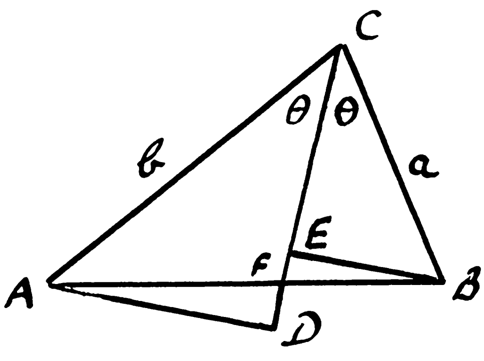
$$\text{Area of triangle } CAB = \tfrac{1}{2}\cdot ab\cdot\sin 2\theta = ab\cdot\sin\theta\cdot\cos 0 \,.$$

Therefore

$$\tfrac{1}{2}\cdot CF\cdot\sin\theta\cdot(a+b) = ab\cdot\sin\theta\cdot\cos\theta \,.$$

Hence

$$CF = \frac{2\cdot ab\cdot\cos\theta}{a+b} \,.$$



Also solved by Brother U. Alfred, St. Mary's College, California; Rodney D. Arner, Chaffey College, California; Merrill Barneby, University of North Dakota; Dermott A. Breault, Sylvania Applied Research Laboratory, Waltham, Massachusetts; Brother T. Brendan, St. Mary's College, California; Maurice Brisebois, University Sherbrooke, Canada; E. F. Canaday, Meredith College, North Carolina; R. T. Coffman, Richland, *Washington; John J. Fisher, Johnstown College, Pennsylvania; P. D. Goodstein, University of Leicester, England; Connie Jarrett, Howard University, Washington D. C.; Richard Levitt, Boston Latin School; James W. Mellender, University of Wisconsin; George Millman, Fort Monmouth, New Jersey; C. C. Oursler, Southern Illinois University; Garth Peterson, Sioux Falls, South Dakota; C. F. Pinzka, University of Cincinnati; R. Ransom, Montreal, Canada; Lawrence A. Ringenberg, Eastern Illinois University; David L. Silverman, Beverly Hills, California; David R. Simpson, University of Alaska; Joseph L. Stearn, Washington, D. C.; Paul Stygar, Yale University; P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.; C. W. Trigg, Los Angeles City College; W. C. Waterhouse, Harvard University; Hazel S. Wilson, Jacksonville University, Florida; Dale Woods, State Teachers College, Kirksville, Missouri; and the proposer (three solutions).*

Prime Products

455. [September 1961] *Proposed by Leonard Carlitz, Duke University.*

Let $n > 1$. Show that:

a) $$x(x+1)\cdots(x+n-1) \equiv x^n - x \pmod{n}$$

if and only if n is prime;

b) $$\prod_{\substack{a=1\\(a,n)=1}}^{n} (x+a) = x^{\phi(n)} - 1 \pmod{n}$$

if and only if $n = p$ or $2p$, where p is a prime.

Solution by the proposer.

a) The sufficiency is familiar. To prove the necessity we observe that a) implies $(n-1)! \equiv -1 \pmod{n}$, which in turn implies that n is prime.

b) Again the sufficiency is well known. To prove the necessity we note first that b) implies

$$\prod_{\substack{a=1\\(a,n)=1}}^{n} a \equiv -1 \pmod{n},$$

which (by the generalized Wilson theorem) requires that $n = 2,\ 4,\ p^r$ or $2p^r$, where p is an odd prime and $r \geq 1$. If $r > 1$ then by Bauer's congruence

$$\prod_{\substack{a=1\\(a,n)=1}}^{n} (x+a) \equiv (x^{p-1}-1)^{p^{r-1}} \pmod{p^r}.$$

For $r > 1$ we have

$$(x^{p-1}-1)^{p^{r-1}} \not\equiv x^{\phi(p^r)} - 1 \pmod{p^r}$$

and the stated result follows.

An Almost Fermat Equation

456. [September 1961] *Proposed by M. S. Klamkin, AVCO, Wilmington, Massachusetts.*

Determine two parameter solutions of the following "almost" Fermat Diophantine equations:

(1) $$x^{n-1} + y^{n-1} = z^n$$

(2) $$x^{n+1} + y^{n+1} = z^n$$

(3) $$x^{n+1} + y^{n-1} = z^n \ .$$

Solution by Leo Moser, University of Alberta. We will exhibit two parameter solutions for the more general equation

(1) $$x^a + y^b = z^c \quad (a,\ b,\ c) = 1\ .$$

Since $(a, b, c) = 1$ we can first find an m and n such that

(2) $$abm + 1 = cn\ .$$

Now let u and v be arbitrary integers and let

(3) $$x = u^{bm}(u^{abm} + v^{abm})^{bm}$$

and

(4) $$y = v^{am}(u^{abm} + v^{abm})^{am}\ .$$

Then

(5) $$x^a + y^b = (u^{abm} + v^{abm})^{abm+1}\ .$$

By (2) we have

$$x^a + y^b = (u^{abm} + v^{abm})^{nc}$$

so that with $z = (u^{abm} + v^{abm})^n$ equation (1) is satisfied.

Also solved by Leonard Carlitz, Duke University; Daniel I. A. Cohen, Philadelphia, Pennsylvania; R. L. Goodstein, University of Leicester, England; J. A. H. Hunter, Toronto, Canada; Aaron Lieberman, Merrick, New York; Richard Levitt, Boston Latin School; Lawrence A. Ringenberg, Eastern Illinois University; David L. Silverman, Beverly Hills, California; and the proposer.

A Ubiquitous Square Root

457. [September 1961] *Proposed by C. W. Trigg, Los Angeles City College.*

In a certain integer, the units' digit 6 is preceded by $(k-1)$ 5's, which in turn are preceded by k 1's. Find the square root of the integer.

Solution by C. F. Pinzka, University of Cincinnati.

I. Denoting the integer by N, we have

$$N = \frac{10^k(10^k - 1)}{9} + \frac{50(10^{k-1} - 1)}{9} + 6 = \frac{(10^{2k} + 4\cdot 10^k + 4)}{9}$$

and

$$\sqrt{N} = \frac{(10^k + 2)}{3} \quad ,$$

which has a units' digit 4 preceded by $(k-1)$ 3's.

II. Let a subscript denote the number of repetitions of a digit. Then $N = 1_k 5_{k-1} 6$. We write

$$\begin{aligned} 10N &= 11_{k-1}55_{k-2}60 \\ N &= 1_{k-1}15_{k-2}56 \\ 9N &= 10_{k-1}40_{k-2}04 \\ &= 10_{k-1}40_{k-1}4 \\ 3\sqrt{N} &= 10_{k-1}2 \\ \sqrt{N} &= 3_{k-1}4 \,. \end{aligned}$$

Also solved by Brother U. Alfred, St. Mary's College, California; Merrill Barneby, University of North Dakota; Joseph B. Bohac, St. Louis, Missouri; Dermott A. Breault, Sylvania Applied Research Laboratory, Waltham, Massachusetts, Maurice Brisebois, University of Sherbrooke, Canada; Maxey Brooke, Sweeney, Texas; Daniel I. A. Cohen; Philadelphia, Pennsylvania; D. Drasin, Temple University; Ralph Greenberg, Philadelphia, Pennsylvania; John M. Howell, Los Angeles City College; Gilbert Labelle, Collège de Longueuil, Canada; Richard Levitt, Boston Latin School; Leo Moser, University of Alberta, Canada; C. C. Oursler, Southern Illinois University; R. Ransom, Montreal, Canada; David L. Silverman, Beverly Hills, California; Paul Stygar, Yale University; W. C. Waterhouse, Harvard University; Dale Woods, State Teachers College, Kirksville, Missouri; and the proposer.

Bohac pointed out that other integers have interesting squares:

$$(\overbrace{9\cdots9}^{k}\,1)^2 = \overbrace{9\cdots9}^{k-1}82\overbrace{0\cdots0}^{k-1}81$$

$$(\overbrace{9\cdots9}^{k}\,2)^2 = \overbrace{9\cdots9}^{k-1}84\overbrace{0\cdots0}^{k-1}64 \,.$$

Moser made a related comment dealing with

$$\begin{aligned} 6^2 - 5^2 &= 11 \\ 56^2 - 45^2 &= 1111 \\ 556^2 - 445^2 &= 111111 \\ &\cdots \end{aligned}$$

and its generalization to any even base.

De Moivre's Theorem

458. [September 1961] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

A student used DeMoivre's theorem incorrectly as

$$(\sin\alpha + i\cos\alpha)^n = \sin n\alpha + i\cos n\alpha \,.$$

For what values of α does the equation hold for every integer n?

Solution by Dermott A. Breault, Sylvania Applied Research Laboratory, Waltham, Massachusetts. Let

$$z = \cos\theta + i\sin\theta \,.$$

Then using DeMoivre's Theorem correctly we have

$$z^n = \cos n\theta + i\sin n\theta \,.$$

The proposed relation is that $(i/z)^n = (i/z^n)$ which implies that

$$(1/z^n)(i^n - i) = 0 \,.$$

But $z^{-n} \neq 0$, so there are no values of θ for which the proposal holds for every integer n, but it is an identity for all n of the form $n = 4k+1$.

Also solved by Brother U. Alfred, St. Mary's College, California; Leonard Carlitz, Duke University; Alan B. Delfino, St. Mary's College, California; P. D. Goodstein, University of Leicester, England; Harvey H. Green, R. C. A. Ascension Island (partially); Richard Levitt, Boston Latin School; David L. Silverman, Beverly Hills, California; Paul Stygar, Yale University; W. C. Waterhouse, Harvard University; and the proposer.

A Double Sum

459. [September 1961] *Proposed by H. M. Gandhi, Lingraj College, Belgaum, India.*

Sum the series

$$\sum_{n=1}^{\infty}\left[\frac{n}{x^n} + \frac{n+1}{x^{n+1}} + \frac{n+2}{x^{n+2}} + \cdots\right] .$$

Solution by David Chale, University of California, Berkeley. Noting that

$$\sum_{n=k}^{\infty}\frac{1}{x^n} = \frac{1}{x^{k-1}(x-1)}$$

we see that

$$\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{1}{x^n}=\frac{x}{(x-1)^2}$$

Then differentiating both sides we have

$$\frac{1}{x}\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{n}{x^n}=\frac{x+1}{(x-1)^3}$$

which is equivalent to

$$\sum_{n=1}^{\infty}\left[\frac{n}{x^n}+\frac{n+1}{x^{n+1}}+\cdots\right]=\frac{x(x+1)}{(x-1)^3}.$$

Also solved by Dermott A. Breault and Richard Michaud, Sylvania Applied Research Laboratory, Waltham, Massachusetts; Maurice Brisebois, University of Sherbrooke, Canada; J. L. Brown, Jr., Pennsylvania State University; Leonard Carlitz, Duke University; P. D. Goodstein, University of Leicester, England; Roger D. H. Jones, College of William and Mary, Virginia; Lawrence A. Ringenberg, Eastern Illinois University; David L. Silverman, Beverly Hills, California; Paul Stygar, Yale University; W. C. Waterhouse, Harvard University; and the proposer.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 294. Show that the vector expression

$$A\times(\nabla\times B)+B\times(\nabla\times A)+(A\cdot\nabla)B+(B\cdot\nabla)A$$

would be the same in an English or an Israeli (reading from right to left) article. [*Submitted by M. S. Klamkin.*]

Q 295. Determine the area of an ellipse with semi-major axis a and semi-minor axis b. [*Submitted by C. W. Trigg.*]

Q 296. If two ellipsoids have an ellipse in common, all their other points of intersection, if real, lie on another ellipse. [*Submitted by M. S. Klamkin.*]

TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase, or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 52. A student writes $\sin(x+y)\sin(x-y)=(\sin x+\sin y)(\sin x-\sin y)$. For what values of x and y is the statement true? [*Submitted by C. F. Pinzka.*]

T 53. Under what circumstances can the diagonals of a quadrilateral be parallel to each other? [*Submitted by Norman Anning.*]

(Answers to Quickies and Solutions for Trickies are on page 126.)

ANSWERS *to Quickies on page 125.*

A 293. [Omitted from Vol. 35, No. 1.] Yes. The number is equal to

$$\frac{1}{9}\cdot\frac{10^9-1}{10-1} = (37)(333{,}667)\ .$$

A 294. Both expressions are just expressions of $\nabla(A\cdot B)$.

A 295. An ellipse with semi-minor axis b is a plane section of a right circular cylinder with radius b. A plane through the center of the ellipse perpendicular to the axis of the cylinder will intersect the cylinder in a circle with radius b. A plane through the axis of the cylinder and the major axis of the ellipse will intersect the other two planes and the surface of the cylinder in a right triangle with hypotenuse a and base b. All planes parallel to this plane will intersect these three surfaces in similar right triangles. Therefore, the area of the ellipse is to the area of the circle as a is to b. That is, $\Delta = \pi b^2(a/b) = \pi ab$.

A 296. Let the equations of the common ellipse be

$$C \equiv ax^2 + bxy + cy^2 + dx + ey + f = 0\ , \quad z = 0\ .$$

The most general equations of two ellipsoids which pass through this ellipse are

$$C + z(a_1x + b_1y + c_1z + d_1) = 0\ ,$$
$$C + z(a_2x + b_2y + c_2z + d_2) = 0\ .$$

All points on both ellipsoids which are not on $z = 0$ must satisfy

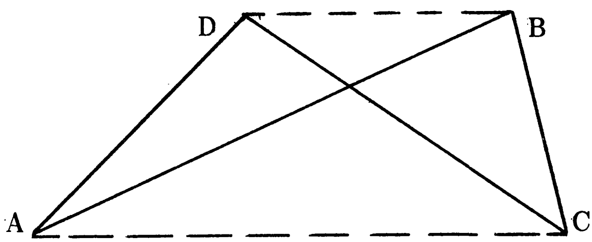
$$a_1x + b_1y + c_1z + d_1 = a_2x + b_2y + c_2z + d_2$$

which is another plane intersecting in another ellipse.

SOLUTIONS *for Trickies on page 125.*

S 52. All values of x and y, since the statement is an identity.

S 53. The quadrilateral must be a "crossed" plane quadrilateral as in the figure where AC and DB are the diagonals.



ERRATUM

In the solution **S 48** on page 436, Vol. 34, No. 7, November 1961, in the last line $\pi - x$ should read πx.

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